MAT165: PROBLEMS FOR PRACTICE

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(1) Let $X = \{a_1, a_2, a_3, a_4, a_5\}$ be a subset of the set of integers which are perfect squares. Show that there exists a subset Y of X, such that for $Y = \{b_1, b_2, b_3\}$ we have $3|(b_1 + b_2 + b_3)$.

Solution. We analyze the residues of perfect squares modulo 3. For any integer n, $n^2 \equiv 0$ or $1 \pmod 3$. Thus, every element in X is congruent to either 0 or 1 modulo 3.

We have 5 elements in X. By the Pigeonhole Principle, when distributing these 5 elements into the 2 possible residue classes (0 and 1):

- We must have at least three elements congruent to $0 \pmod 3$, OR
- We must have at least three elements congruent to $1 \pmod 3$.

Case 1: There are 3 elements b_1, b_2, b_3 such that $b_i \equiv 0 \pmod 3$. Then $b_1 + b_2 + b_3 \equiv 0 + 0 + 0 \equiv 0 \pmod 3$.

Case 2: There are 3 elements b_1, b_2, b_3 such that $b_i \equiv 1 \pmod 3$. Then $b_1 + b_2 + b_3 \equiv 1 + 1 + 1 \equiv 3 \equiv 0 \pmod 3$.

In both cases, the sum is divisible by 3. Thus, such a subset Y always exists.

(2) A is a 51 element subset of $\{1, 2, \ldots, 100\}$ such that no two numbers from A add upto 100, show that A contains a square.

Solution. We partition the set $\{1, 2, \ldots, 100\}$ into disjoint sets based on the condition $x + y = 100$:

- 49 pairs: $\{1, 99\}, \{2, 98\}, \ldots, \{49, 51\}$.
- 2 singletons: $\{50\}$ and $\{100\}$ (since $50 + 50 = 100$ requires two 50s, and 100 requires 0).

To form a subset A where no two numbers sum to 100, we can select at most 1 number from each of the 49 pairs. This gives a maximum of 49 elements. To reach the required size of 51 elements, we are forced to select the remaining available numbers: the singletons $\{50\}$ and $\{100\}$.

Thus, $100 \in A$. Since $100 = 10^2$, the set A contains a perfect square.

(3) What is the maximum number of non-attacking bishops that you can place on a $n \times n$ chessboard?

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Solution. The maximum number is $2n-2$. *Proof Sketch:* Bishops attack along diagonals. We can verify the bound by placing bishops on the outer edges of the board, excluding the two opposite corners that share a long diagonal. A valid configuration is:

$$\{(1,1),\ldots,(1,n-1)\} \cup \{(n,1),\ldots,(n,n-1)\}$$

This gives $(n-1)+(n-1)=2n-2$ bishops.

(4) Suppose the vertices of a regular polygon of 20 sides are colored with 3 colours, say R, B and G such that there are exactly 3 vertices of the colour R. Prove that there are 3 vertices of the polygon having the same colour such that they form an isoceles triangle.

Solution. Let the vertices be V. We are given 3 vertices colored Red (R).

- If the 3 R vertices form an isosceles triangle, we are done.
- If not, consider the remaining $20-3=17$ vertices. These must be colored Blue (B) or Green (G).

By the Pigeonhole Principle, distributing 17 vertices into 2 colors implies at least one color (say B) has $\lceil 17/2 \rceil = 9$ vertices.

Now form four disjoint pentagons out of the 20 vertices, by PHP again, there must be one pentagon where we have at least 3 vertices of the same colour, which gives us the required isoceles triangle.

(5) Show that the numbers 1 to 81 cannot be arranged in a 9×9 chessboard so that the product of the entries in row i equals the product of the entries in column j for some j, such that $1\le j\le 9$.

Solution. Assume for contradiction that the product of Row i ($P(R_i)$) equals the product of Column j ($P(C_j)$). Consider the prime numbers p such that $41\le p\le 81$. These are $\{41,43,47,53,59,61,67,71,73,79\}$. There are exactly 10 such primes.

In the set $\{1,\ldots,81\}$, each of these primes appears exactly once (since $2\times 41 = 82 > 81$). For $P(R_i)=P(C_j)$, any prime factor appearing in the row product must also appear in the column product. If a large prime p is in Row i, it must be in Column j for the products to be equal. Since p appears only once on the whole board, the number containing p must be placed at the intersection (i,j).

This logic applies to **all** such large primes present in Row i. By the Pigeonhole Principle, since there are 10 large primes and 9 rows, at least one row must contain two large primes, say p_1 and p_2. For the row/column products to match, both p_1 and p_2 must be at the intersection cell (i,j). This implies the number at (i,j) is a multiple of p_1p_2. However:

$$p_1\cdot p_2 \ge 41\cdot 43 = 1763 > 81$$

This contradicts the fact that entries are ≤ 81. Thus, such an arrangement is impossible.

(6) In a row of 35 chairs find the minimum number of chairs that must be occupied such that there are some consecutive set of 4 or more occupied chairs.

Solution. Let $n = 35$. We want to find the minimum k occupied chairs that forces a block of 4. This is equivalent to finding the maximum number of chairs we can occupy *without* creating a block of 4, then adding 1. To avoid 4 consecutive chairs, we can use a repeating pattern of 3 occupied (O) and 1 empty (E): $OOOE$.

The pattern length is 4. We fit as many patterns as possible:

$$35 = 8 \times 4 + 3$$

We can fit 8 blocks of $OOOE$, followed by 3 occupied chairs OOO. Max occupied without 4 consecutive:

$$8 \times 3(\text{from blocks}) + 3(\text{remainder}) = 24 + 3 = 27$$

Therefore, if we occupy $27+1 = 28$ chairs, we are forced to have 4 consecutive occupied chairs.

(7) What is the largest number of squares on an 8×8 board which can be coloured green so that in any tromino, at least one square is not coloured green.

Solution. Let Green (G) be the colored squares and White (W) be the uncolored. We want to maximize G, which implies minimizing W such that every tromino contains at least one W.

Lower Bound for W: We can tile an 8×8 board (64 squares) with disjoint trominoes. Since $64 = 21 \times 3 + 1$, we can place 21 disjoint trominoes. Each must contain at least one W. Therefore, we need at least 21 W squares. Max $G = 64 - 21 = 43$.

Construction: We can achieve this by leaving square (i, j) uncolored (White) if $i + j \equiv 1 \pmod 3$. This ensures every horizontal tromino $\{(i, j), (i, j + 1), (i, j + 2)\}$ and vertical tromino $\{(i, j), (i + 1, j), (i + 2, j)\}$ contains exactly one square where the sum of indices is $\equiv 1 \pmod 3$. Counting these squares yields exactly 21.

(8) In a state there are 100 cities and 4 roads lead out of every city. How many roads are there in total?

Solution. Let V be the number of cities and E be the number of roads. Given: $V = 100$, and degree of each vertex (the number of roads coming out of every city) $\deg(v) = 4$. Try to show the following is true

$$2E = \deg(v) \times V.$$