MAT165 ASSIGNMENT 2

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Last Date of Submission. 07 November, 2025 **before** the lecture starts.

Instructions.

- You can discuss the problems with any of your class-mates. In fact, I encourage you to talk to your friends and come up with the solutions together.
- Avoid using AI or web search to arrive at the solutions. This way would be easy, but you will learn very little.
- Write down the solutions in A4 sized sheets of paper (either blank or dotted), and staple them before submission. Use either black or blue ink for writing the solutions.
- If you are unable to solve a problem, then mention what approach you took and how that did not work out. Sometime in mathematics, false starts can lead to promising avenues in other directions.

Questions.

(1) Suppose you have a white box with 60 white balls and a black box with 60 black balls. You take 20 balls from the white box and put them into the black box and mix everything up. Next you take 20 balls from the black box and put them into the white box. In the end, which is larger: the number of black balls in the white box or the number of white balls in the black box? Explain your reasoning.

(2) Suppose that a white cup contains some milk and a black cup contains the same amount of coffee. We take 3 spoons of milk from the white cup and pour into the black cup, stir the black cup thoroughly and then pour 3 spoons of this mixture into the white cup. Which is the higher ratio: coffee to milk in white cup or milk to coffee in black cup? How does the answer change if you repeat this process once again? Explain your reasoning.

(3) Prove that there is no smallest positive rational number.

(4) If every room in a house has an even number of doors, then show that the number of door which open outside must be even as well.

(5) Conjecture the largest natural number which is guaranteed to divide $n^3 - n$ for all $n > 0$. Then, prove your conjecture using the principle of mathematical induction (PMI).

(6) Conjecture a formula for the sum

$$\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{n(n+1)}, \quad n \geq 1.$$

Show how you arrived at your conjectured formula. Finally, prove the formula using PMI.

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What about a formula for

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\cdots+\frac{1}{(n-1)n}, \quad n\geq 2?$$

(7) What is wrong with the following proof that $a^n = 1$ for all integers $n \geq 0$ and real numbers $a \neq 0$?

Base Case. For $n = 0$, we have $a^0 = 1$. So, the base case holds.

Induction Hypothesis. We assume that $a^k = 1$ holds for some $k \geq 0$.

Induction Step. We notice that $a^{k+1} = a^k \cdot a^1 = a^k \cdot \dfrac{a^k}{a^{k-1}} = 1 \cdot \dfrac{1}{1}$.

(8) Show that

$$\frac{1}{a(a+b)}+\frac{1}{(a+b)(a+2b)}+\cdots+\frac{1}{(a+nb-b)(a+nb)}=\frac{n}{a(a+nb)},$$

where a, b are natural numbers.

Is there a connection between this problem and problem (6) above?

(9) It is allowed to tear a piece of paper into 4 or 6 smaller pieces. Prove that following this rule you can tear a sheet of paper into any number of pieces greater than 8.