MAT165 ASSIGNMENT 5

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Instructions.

- You can discuss the problems with any of your class-mates. In fact, I encourage you to talk to your friends and come up with the solutions together.
- Avoid using AI or web search to arrive at the solutions. This way would be easy, but you will learn very little.
- Write down the solutions in A4 sized sheets of paper (either blank or dotted), and staple them before submission. Use either black or blue ink for writing the solutions. **Failure to follow this will result in an immediate score of zero.**
- If you are unable to solve a problem, then mention what approach you took and how that did not work out. Sometime in mathematics, false starts can lead to promising avenues in other directions.

Questions.

(1) Consider a set of $N = 12$ discrete tokens, $\mathcal{C}$, where each token c_i assumes a binary state $S \in \{H, T\}$ (Heads or Tails). Let the initial configuration of states, defined by the count of heads and tails, be mutually known to two independent agents, A and B. Agent B executes an arbitrary, finite sequence of k operations, where $k \geq 0$. An operation $\mathcal{O}$ consists of selecting a single token $c \in \mathcal{C}$ and inverting its state ($H \leftrightarrow T$). Agent A is unaware of the specific tokens selected but is supplied with the total count k of operations performed. Upon the conclusion of B's actions, A is presented with the final states of a specific subset of $N - 1 = 11$ tokens. Does A possess sufficient information (the initial state, the total count k, and the observed 11 states) to uniquely determine the final state of the single unobserved token? Provide a rigorous justification based on the underlying algebraic invariant.

(2) The integers $1, 2, \ldots, n$ are written in a circle. In one operation, you can choose any two adjacent numbers a and b and swap them, but only if the difference $|a - b|$ is odd. Determine all values of n for which it is possible to rotate the entire circle of numbers

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one position clockwise (i.e., transform the initial arrangement $(1, 2, 3, \ldots, n)$ into the arrangement $(n, 1, 2, \ldots, n-1)$) using a finite sequence of these operations.

(3) A large rectangular box has dimensions $10 \times 10 \times 10$. We wish to pack the box completely using "bricks" of size $1 \times 1 \times 4$. Bricks may be oriented in any of the three axial directions, but they must not overlap and must be contained entirely within the box. Prove that it is impossible to fill the box exactly with these bricks.

(4) A disk is divided into $2k$ equal sectors, where k is an odd integer. In each sector, a coin is placed. Initially, all $2k$ coins are showing "heads." In one move, you are permitted to choose any k consecutive sectors and flip every coin within those sectors (heads becomes tails, and tails becomes heads). Is it possible, through a sequence of such moves, to reach a state where exactly one coin is showing "heads" and all other $2k-1$ coins are showing "tails"? Justify your answer.

(5) An infection spreads on an $n \times n$ checkerboard. Initially, some squares are "infected" (colored black), while others are healthy (colored white). The infection spreads according to the following rule: a healthy square becomes infected if at least two of its orthogonal neighbors (up, down, left, or right) are already infected. Once a square is infected, it remains infected forever. Prove that if the entire $n \times n$ board eventually becomes infected, then there must have been at least n infected squares in the initial configuration.

(6) The numbers $1, 2, 3, \ldots, 2024$ are written on a blackboard. In each step, you are permitted to choose any two numbers a and b currently on the board, erase them, and replace them with a single new number calculated as $a + b + ab$. This process is repeated until only one number remains on the board. Prove that the value of this final number is independent of the sequence of choices made, and find its value.