MAT215 ELEMENTARY NUMBER THEORY & CRYPTOGRAPHY ASSIGNMENT 1

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Instructions.

- Due: before the lecture on **19th September 2025**.
- Must be submitted as a PDF (via email) or printed/written (slide under my office door).
- Begin each solution on a new page, and use only blank or dotted pages (not ruled).
- Staple the submission, if submitting a paper version.
- State your sources at the top of each problem (even if you worked independently).
- Late penalty is 25% per late day.
- Failure to follow any of these instructions will result in 0 points.

Any letter denoting a number (such as a, b, c, n, etc.) is always considered to be an integer, unless otherwise mentioned.

Problems. All problems are mandatory.

The following problems are worth 1 point each.

(1.1) If $n > 2$, prove that there exists a prime p such that $n < p < n!$.

(1.2) If p_i is the i-th prime number, then show that $p_1p_2\cdots p_n + 1$ is never a perfect square.

(1.3) If p and p^2+8 are both primes, then show that p^3+4 is also a prime.

(1.4) Show that $\gcd(n!+1, (n+1)!+1) = 1$ for any $n \in \mathbb{Z}^+$.

The following problems are worth 2 points each.

(2.1) Show that every positive integer n can be uniquely represented in the form

$$n = n_0 + 3n_1 + 3^2n_2 + \cdots + 3^tn_t$$

with $n_i \in \{0, 1, -1\}$ for $0 \le i \le t$ where $t \ge 0$ is an integer.

(2.2) Show that for every positive integers $n \ge m$,

$$\frac{\gcd(m,n)}{n}\binom{n}{m}$$

is always an integer.

(2.3) The prime numbers p and q are called *twin primes* if $|p-q| = 2$. Prove that $pq+1$ is a square if and only if p and q are twin primes.

(2.4) (a) If the product of 2002 integers equals 1, can their sum be zero?

(b) What if you replace 2002 by any positive even integer?

(2.5) Show that $61! + 1 \equiv 63! + 1 \equiv 0 \pmod{71}$.

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(2.6) Find the smallest positive integer n for which $2^2 \mid n$, $3^2 \mid (n+1)$, $5^2 \mid (n+2)$ and $7^2 \mid (n+3)$.

(2.7) A basket contains n mangoes. If it is distributed among 2 people, 1 mango is left. If we distribute among $3, 5$ or 7 people, we are left with $2, 4$ or 6 mangoes, respectively. However we can distribute equally among 11 people. Find the smallest such n.

(2.8) Given a positive integer N, let $\overline{N}$ be the positive integer obtained by writing digits of N in reverse order. For example, if $N = 1234$, then $\overline{N} = 4321$ and if $N = 1230$, then $\overline{N} = 321$. Find the number of all N such that $N - \overline{N} = 9$.