MAT215 ELEMENTARY NUMBER THEORY & CRYPTOGRAPHY PROBLEM SHEET 2

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Any letter denoting a number (such as a, b, c, n, etc.) is always considered to be an integer, unless otherwise mentioned.

(1) Show that the expression $\dfrac{(2n)!}{2^n n!} \in \mathbb{Z}$, for all $n \geq 0$.

(2) For $2 \leq k \leq n-2$, show that

$$\binom{n}{k} = \binom{n-2}{k-2} + 2\binom{n-2}{k-1} + \binom{n-2}{k}, \quad n \geq 4.$$

(3) Show that the cube of any integer is of the form $7k$ or $7k \pm 1$.

(4) If n is an odd integer, show that $n^4 + 4n^2 + 11$ is of the form $16k$.

(5) For any odd integer, what is $\gcd(3a, 3a+2)$?

(6) Prove that $360 | a^2(a^2-1)(a^2-4)$, for any integer a.

(7) Prove that if $\gcd(a, b) = 1$, then $\gcd(a+b, ab) = 1$.

(8) Solve the Diophantine equation: $158x - 57y = 7$ for positive values of x and y.

(9) Prove that if $n > 4$ is composite, then $n | (n-1)!$.

(10) If $p \geq 5$ is a prime number, then show that $p^2 + 2$ is composite.

(11) Using the theory of congruences show that $7 | (5^{2n} + 3 \cdot 5^{2n-2})$ for all $n \geq 1$.

(12) If p is a prime such that $n < p < 2n$, then show that $\binom{2n}{n} \equiv 0 \pmod{p}$.

(13) Show that $89 | (2^{44} - 1)$ and $97 | (2^{48} - 1)$.

(14) Find the remainder when 4444^{4444} is divided by 9.

(15) Find the last three digits of 7^{999}. (Hint: Look at $7^{4n} \pmod{1000}$.)

(16) Solve: $34x \equiv 60 \pmod{98}$.

(17) What is the last digit of $12345678910111213141516171819^5$?

(18) Find the unit digit of 3^{100} using FLT.

(19) Solve: $x^2 + 1 \equiv 0 \pmod{19}$.

(20) Find one solution of $x^2 + 1 \equiv 0 \pmod{13}$.

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