MAT631 PROBLEM SET 2

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Last Date of Submission. 31 January, 2025.

Instructions. Please read the instructions on the course website carefully before submitting your solution(s).

Questions.

(1) **(Shubham)** Complete the second case (which was skipped in the lecture) in the proof of Euler's pentagonal number theorem.

(2) **(Anant)** Deduce the recursion for partitions mentioned in the lecture, that follows as a corollary of Euler's pentagonal number theorem.

(3) **(Saikat)** Prove the q-binomial theorem as stated in the lecture.

(4) **(Anubhav)** Consider tilings of an $1 \times \infty$ board using white squares, black squares, and white dominoes (that is, a 1×2 board). The weighing scheme for this type of tilings is as follows: if the tile is a black square in position i, the weight of that tile is aq^i, if it is a domino covering positions i and $i+1$ then the weight of the tile is bq^i, and it is 1 for a white square irrespective of the position. In any valid tiling, we allow a finite number of black squares and dominoes. If the weight of a tiling is just the product of the weights of the tiles in that tiling, show that the generating function for such tilings is given by

$$F_q(a,b) := \sum_{n\geq 0} \frac{(a+b)(a+bq)\cdots(a+bq^{n-1})}{(q;q)_n} q^{n(n+1)/2}.$$

(5) **(Kanak)** Recall the definitions from the previous problem. Prove the following formulas (combinatorially)

(a) $F_q(a,b) = F_q(aq,b) + aqF_q(aq,bq)$, and

(b) $F_q(a,b) = F_q(q,bq) + bqF_q(aq,bq^2)$.

Date: 23 January 2025.