Solving System of Simultaneous Equations using Matrices

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System of linear equations

A **System of m linear equations** in n **variables** $x_1, x_2, \ldots, x_n$ is a collection of m equations of the following form

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m. \end{aligned}$$

- The numbers a_{ij} are called the **coefficients** of the system of linear equations.
- We have to find to all the solutions of this system, a **solution** is a list of n numbers, say $c_1, c_2, \ldots, c_n$ that satisfy the system of linear equations. The set of all solutions is called the **solution set**.

Types of systems

- A system may not have any solution, in this case we say that the system is **inconsistent**.
  - For example, the system $x - y = 1$ and $y - x = 3$ has no solutions in natural numbers.
- If a system has at least one solution then we say that the system is **consistent**.
  - For example, the system $x - y = 3$ and $2x - y = 7$ has one solution in natural numbers, namely $(x, y) = (4, 1)$.
- A consistent linear system will have either just one solution or infinitely many solutions.
  - For example, the system $x_1 + x_2 + x_3 = 0$ and $x_1 + 3x_2 - x_3 = 3$ has infinitely many solutions. $(x_1, x_2, x_3) = (-3/2 - 2r, 3/2 + r, r)$ for any $r \in \mathbb{R}$.

Matrices

A **matrix** is an array or table consisting of rows and columns.

$$M = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 1 & 3 & -1 & 3 \end{pmatrix}$$

If a matrix has m rows and n columns then we say that such a matrix is an $m \times n$ matrix. The matrix M is a 2×4 matrix.

A matrix containing only one column is called a **column vector** and a matrix containing only one row is called a **row vector**.

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 & 1 & 0 \end{pmatrix}$$

Matrices and Systems of equations

We can associate a linear system with three matrices

- the coefficient matrix,
- the output column vector, and
- the augmented matrix.

For example, take the system

$$\begin{aligned} -3x_1 + 2x_2 + 4x_3 &= 12 \\ x_1 - 2x_3 &= -4 \\ 2x_1 - 3x_2 + 4x_3 &= -3. \end{aligned}$$

Here the matrices are respectively

$$A = \begin{pmatrix} -3 & 2 & 4 \\ 1 & 0 & -2 \\ 2 & -3 & 4 \end{pmatrix}, b = \begin{pmatrix} 12 \\ -4 \\ -3 \end{pmatrix}, (A|b) = \left(\begin{array}{ccc|c} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \\ 2 & -3 & 4 & -3 \end{array}\right).$$

Solving equations via Matrices

We can perform three basic operations, called **elementary operations** on a system of linear equations:

- interchange two equations,
- multiply an equation by a nonzero constant, and
- add a multiple of one equation to another.

These operations do not alter the solution set!

- In terms of of the augmented matrix representing the linear system we call the operations **elementary row operations**.
- The goal with row reducing is to transform the original linear system into one having a triangular structure and then we perform back substitution to solve the system.

An example

Let us solve the following system with the above technique:

$$\begin{aligned} x_1 + 2x_2 - 3x_3 &= 2 \\ x_2 + 2x_3 &= 10 \\ x_3 &= 3 \end{aligned}$$

The augmented matrix is

$$\left(\begin{array}{ccc|c} 1 & 2 & -3 & 2 \\ 0 & 1 & 2 & 10 \\ 0 & 0 & 1 & 3 \end{array}\right)$$

From the last equation we get $x_3 = 3$, substituting this in the second equation we get $x_2 + 6 = 10$, giving us $x_2 = 4$. And finally, the first equation gives us $x_1 + 8 - 9 = 2$, giving us $x_1 = 3$.

Another example

Let us go back to our earlier example

$$
\begin{aligned}
-3x_1 + 2x_2 + 4x_3 &= 12 \\
x_1 - 2x_3 &= -4 \\
2x_1 - 3x_2 + 4x_3 &= -3.
\end{aligned}
$$

Here the augmented matrix is

$$
\left(\begin{array}{ccc|c} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \\ 2 & -3 & 4 & -3 \end{array} \right).
$$

We will now use elementary row operations to get the augmented into a triangular structure.

Row Operations

Interchange Row 1 with Row 2:

$$\left(\begin{array}{ccc|c} -3 & 2 & 4 & 12 \\ 1 & 0 & -2 & -4 \\ 2 & -3 & 4 & -3 \end{array}\right) \quad \rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ -3 & 2 & 4 & 12 \\ 2 & -3 & 4 & -3 \end{array}\right)$$

Add $3\times$ Row 1 with Row 2:

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ -3 & 2 & 4 & 12 \\ 2 & -3 & 4 & -3 \end{array}\right) \quad \rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 2 & -3 & 4 & -3 \end{array}\right)$$

Add $-2\times$ Row 1 to Row 3:

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 2 & -3 & 4 & -3 \end{array}\right) \quad \rightarrow \quad \left(\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & 8 & 5 \end{array}\right)$$

Example contd.

As a last step, we now multiply $\frac{3}{2}\times$ Row 2 to Row 3:

$$\left(\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 2 & -2 & 0 \\ 0 & -3 & 8 & 5 \end{array}\right) \quad \to \quad \left(\begin{array}{ccc|c} 1 & 0 & -2 & -4 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 5 & 5 \end{array}\right)$$

Our original system now becomes

$$\begin{aligned} x_1 - 2x_2 &= -4 \\ 2x_2 - 2x_3 &= 0 \\ 5x_3 &= 5 \end{aligned}$$

Via back substitution we get $x_3 = 1$, $x_2 = 1$ and $x_1 = -2$.

If we obtain a row in an augmented matrix in any step which is of the form

$$\left(\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & b \end{array}\right)$$

then the system is **inconsistent**.

Thank you!