Applications of Riemann integrals:

$R := \{(x,y)\in \mathbb{R}^2 : a \le x \le b,\ 0 \le y \le f(x)\}$ ... a region.

$f:[a,b]\to\mathbb{R}$ is a bdd nonnegative fn.

$$\text{Area}(R) := \int_a^b f(x)\,dx.$$

• Let $f_1, f_2 : [a,b]\to\mathbb{R}$ be int. fns s.t. $f_1 \le f_2$. Then the area of the region betn the curves given by $y = f_1(x)$, $y = f_2(x)$ and betn the vertical lines $x=a, x=b$ is defined to be $\int_a^b [f_2(x) - f_1(x)]\,dx$.

The region here is $R := \{(x,y)\in\mathbb{R}^2 ; a \le x \le b,\ f_1(x) \le y \le f_2(x)\}$

• If R can be divided into finite no. of such nonoverlapping subregion then Area (R) is just sum of areas of the subregions.

eg: If curves given by $y = f_1(x)$, $y = f_2(x)$ with $f_1, f_2 : [a,b]\to\mathbb{R}$ are contn and cross each other at a finite no. of pts, then the area of the region bounded by those curves and the lines $x=a, x=b$ is equal to

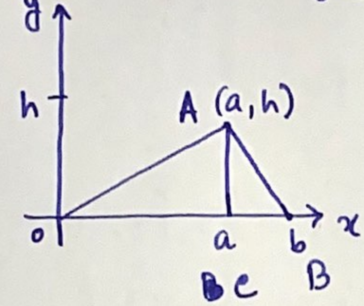
$$\int_a^b |f_2(x) - f_1(x)|\,dx.$$

eg: If $g_1, g_2 : [c,d]\to\mathbb{R}$ are int. such that $g_1 \le g_2$, then the area of the region betn $x = g_1(y)$, $x = g_2(y)$ and the horizontal lines $y=c$, $y=d$ (i.e. the region given by $R := \{(x,y)\in\mathbb{R}^2 : c \le y \le d,\ g_1(y) \le x \le g_2(y)\}$) is defined as,

$$\int_c^d [g_2(y) - g_1(y)]\,dy.$$

Example:

1



$0<a<b$, triangular region enclosed by

$y = hx/a$, $y = h(x-b)/(a-b)$ and $y=0$.

Base $= b$, height $= h$.

Area of $\Delta\, OAB$ = Area of $\Delta\, OAC$ + Area of $\Delta\, ACB$.

base $= a$, height h

base $= b-a$, height $= h$.

The first Δ is bet^n $y = hx/a$, $y=0$ and the lines $x=0$, $x=a$.

So, its area $= \displaystyle\int_0^a \left(\frac{hx}{a} - 0\right) dx = \frac{h}{a}\cdot\frac{a^2}{2} = \frac{ha}{2}$.

IIIly, 2^{nd} Δ ~~is bet~~ has area $\dfrac{h(b-a)}{2}$. Now we add the two.//.

2 The area of the region bounded by the curves $x = y^3$, $x = y^5$ and the lines $y=-1$, $y=1$ is equal to,

$$\int_{-1}^{1} |y^5 - y^3|\, dy = \int_{-1}^{0} (y^5 - y^3)\, dy + \int_0^1 (y^3 - y^5)\, dy .//.$$

Volume of a Solid:

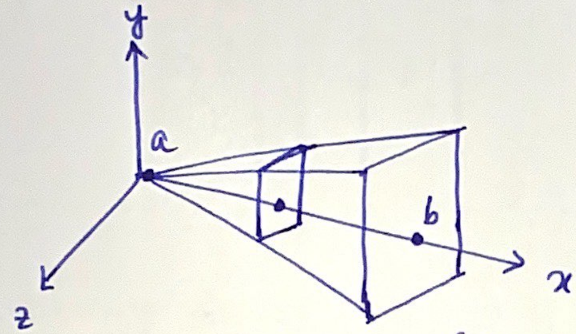
Vol^m of solid bodies can be thought to be made up of cross-sections taken in one of the following ways:

(a) Cross-sections by planes $\perp^r$ to a fixed line,

(b) Cross-section by right circular cylinders having a fixed axis.

We will just focus on the first way in this course.

Let D be a bdd subset of $\mathbb{R}^3$ lying bet^n two parallel planes and let L be a line $\perp^r$ to these planes. A cross section of D by a plane is called a slice of D.



Let L be the x-axis and let D lies bet^n the planes given by $x=a$ and $x=b$, $a,b\in\mathbb{R}$, $a<b$.

For $s\in[a,b]$, let $A(s)$ denote the area of the slice $\{(x,y,z)\in D : x=s\}$ which is obtained by intersecting D with the plane $x=s$. If $(x_0,x_1,\dots,x_n)$ is a partition of $[a,b]$, then D gets divided into n subsolids, $\{(x,y,z)\in D : x_{i-1}\le x\le x_i\}$, $i=1,2,\dots,n$.

Let $s_i\in[x_{i-1},x_i]$ and we replace the i^{th} subsolid by a rectangular slab with vol^m $A(s_i)(x_i-x_{i-1})$. Then $\sum_{i=1}^{n} A(s_i)(x_i-x_{i-1})$ is an approx^m of the vol^m of D.

So, we define $\mathrm{Vol}(D) := \int_a^b A(x)\,dx$, provided the area fn $A:[a,b]\to\mathbb{R}$ is int.

- If $c,d\in\mathbb{R}$, $c<d$, $D\subseteq\{(x,y,z)\in\mathbb{R}^3 : c\le y\le d\}$, for $t\in[c,d]$, $A(t)$ is the slice $\{(x,y,z)\in D : y=t\}$ obtained by intersecting D with the plane $y=t$, then we define

$$\mathrm{Vol}(D) := \int_c^d A(y)\,dy, \text{ provided } A:[c,d]\to\mathbb{R} \text{ is int.}$$

eg: Let $a\in\mathbb{R}$, $a>0$. D be the solid enclosed by the cylinders $x^2+y^2=a^2$ and $x^2+z^2=a^2$. The solid D lies bet^n the planes $x=-a$, $x=a$, and for $s\in[-a,a]$, the slice

$$\{(x,y,z)\in D : x=s\} = \{(s,y,z)\in\mathbb{R}^3 : |y|\le\sqrt{a^2-s^2},\ |z|\le\sqrt{a^2-s^2}\}.$$

The slice is a sq. of side $2\sqrt{a^2-s^2}$ and area $(2\sqrt{a^2-s^2})^2$.

So, $vol(D) = \int_{-a}^{a} A(x)\,dx = 4\int_{-a}^{a} (a^2 - x^2)\,dx$

$$= 8\int_0^a (a^2 - x^2)\,dx \,. //.$$

<u>Solids of Revolution</u>: A subset of $\mathbb{R}^3$ that can be generated by revolving a planar region about an axis is known as a <u>solid of revolution</u>.

If the planar region being revolved is bounded and the axis of revolution is one of the coordinate axis then the vol$^{\underline{m}}$ of the solid of revolution can be found by using ideas of vol$^{\underline{m}}$s of solids.

Let $f_1, f_2 : [a,b] \to \mathbb{R}$ be int. fns s.t. $0 \le f_1 \le f_2$, and suppose the region bet$^{\underline{n}}$ the curves given by $y = f_1(x)$, $y = f_2(y)$ and $x = a$, $x = b$ is revolved arond the x-axis. Let D be the solid of revolution. Then, for $s \in [a,b]$, the area $A(s)$ of the annular slice of D by the plane $x = s$ is equal to $\pi f_2(s)^2 - \pi f_1(s)^2$, so the vol$^{\underline{m}}$ is equal to,

$$vol(D) = \pi\int_a^b [f_2(x)^2 - f_1(x)^2]\,dx \,.$$

<u>eg</u>: $a, h \in \mathbb{R}^+$, a right circular cylindrical solid D of radius a and ht. h is obtained by revolving the rectangular region bdd by $f_2(x) = a$, $f_1(x) = 0$, $x = 0$, $x = h$ about the x-axis. Here, $vol(D) = \pi\int_0^h a^2\,dx = \pi a^2 h$. //.

Arc Length of a curve: A parametrically defined curve C in $\mathbb{R}^2$ given by $(~~f~~x(t), y(t))$, $t \in [a,b]$ is said to be smooth if the fns x and y are diff and their derivatives are cont. on $[a,b]$. In this case, the arc length of C is defined to be $\ell(C) := \int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt$.

In the special case for curves of the form $y = f(x)$ or $x = g(y)$ we have for $f: [a,b] \to \mathbb{R}$, $a < b$, $g: [c,d] \to \mathbb{R}$, $c < d$; $\ell^f(C) = \int_a^b \sqrt{1 + f'(x)^2}\, dx$, $\ell^g(C) := \int_c^d \sqrt{1 + g'(y)^2}\, dy$.

A parametrically defined curve C is said to be piecewise smooth if the fns x and y are cont. on $[a,b]$ and there is a ~~fine~~ partition $(x_0, x_1, \dots, x_n)$ of $[a,b]$ s.t. for each i, the curve given by $(x(t), y(t))$, $t \in [x_{i-1}, x_i]$ is smooth. In this case, $\ell(C) := \sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} \sqrt{x'(t)^2 + y'(t)^2}\, dt$.

eg: Let $\alpha \in \mathbb{R}$, consider $y = ax^2$, $x \in [0,1]$.

Arc length $= \int_0^1 \sqrt{1 + (2ax)^2}\, dx$. (Substitute $u = 2ax$.)

Area of a surface of revolution:

A surface of revolution is generated when a curve is revolved about a line. Let $C := (x(t), y(t))$, $t \in [a,b]$, L be a line in $\mathbb{R}^2$ given by $ax + by + c = 0$, $a, b, c \in \mathbb{R}$.

We just consider when the curve C is a line segment P_1P_2 with $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$. Then, C is given by,

$x(t) := (x_2 - x_1)t + x_1$, $y(t) := (y_2 - y_1)t + y_1$, $t \in [0,1]$.

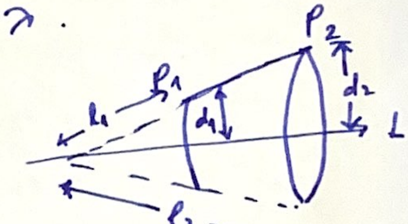
Further assume that the line segment P_1P_2 doesn't cross L. Let d_1 and d_2 be distances of P_1 and P_2 from L. Let λ be the length of P_1P_2.

Note if $P_1P_2 \perp L$, then $\lambda = |d_1 - d_2|$ and the surface of revolution is a circular washer with radii d_1 and d_2.

So, area $= |\pi d_1^2 - \pi d_2^2| = \pi(d_1+d_2)\lambda$.

If $P_1P_2 \parallel L$, then $d_1 = d_2 = d$ (say) and the surface of revolution is a right cylinder with radius d and length λ. Here area is $2\pi d\lambda = \pi(d_1+d_2)\lambda$.

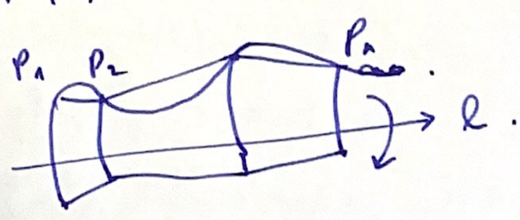
If $P_1P_2 \not\perp L$, $P_1P_2 \not\parallel L$, then the surface of revolution is a frustum (a piece) of a right circular cone with base radii d_1 and d_2 and slant height λ.



In this case as well we can show, area is $\pi(d_1+d_2)\lambda$. (Exercise).

In a general case, when $C := (x(t), y(t))$, a piecewise smooth curve on $(x_0, x_1, \cdots, x_n)$ a partition of $[a,b]$. We replace the pieces $(x(t), y(t))$, $t \in [x_{i-1}, x_i]$ by the line segments $P_{i-1}P_i$, with $P_i := (x(x_i), y(x_i))$. Then the sum of the areas of the frustums of the cones generated by these line segments is $\sum_{i=1}^{n} \pi(d_{i-1}+d_i)\lambda_i$ where d_i is distance of P_i from L and $\lambda_i = |P_{i-1}P_i|$.

This sum is an approx. value of the reqd. area of the surface of revolution.



For each i, we have, $d_i = \dfrac{|ax(t_i) + by(t_i) + c|}{\sqrt{a^2+b^2}}$,

$$\lambda_i = \sqrt{(x(t_i) - x(t_{i-1}))^2 + (y(t_i) - y(t_{i-1}))^2}$$

If x and y are cont. diff on (t_{i-1}, t_i) then by the MVT, $\exists\, s_i, u_i \in (t_{i-1}, t_i)$ s.t. $\lambda_i = \sqrt{x'(s_i)^2 + y'(u_i)^2} \times (t_i - t_{i-1})$.

So we can approximate the ~~sum~~ integral

$$\int_a^b \frac{|ax(t) + by(t) + c|}{\sqrt{a^2+b^2}} \sqrt{x'(t)^2 + y'(t)^2}\, dt$$

by $\sum_{i=1}^{n} d_{i-1}\lambda_i$ and $\sum_{i=1}^{n} d_i \lambda_i$.

Hence we define, 2π times this integral to be the area of the surface of revolution obtained by revolving the curve C about the line L.

eg: Consider the line segment $\frac{x}{a} + \frac{y}{h} = 1$, $x \in [0, a]$, $a, h > 0$. The surface area of the cone S of radius a and height h generated by revolving this line segment about the y-axis is, $2\pi \int_0^a a\left(1 - \frac{y}{h}\right) \sqrt{1 + \left(\frac{a}{h}\right)^2}\, dy$

$$= 2\pi a \frac{\sqrt{a^2+h^2}}{h}(h - h/2) = \pi a \sqrt{a^2+h^2}. //.$$