Continuity of functions:

continuity .... roughly "how much" the value of the fn changes when the argument changes "a little".

Def^n: Given a set $A \subset \mathbb{R}$ and a pt. $x_0 \in \mathbb{R}$, we say that x_0 is a cluster pt. for the set A if in any open interval I containing x_0, there are infinitely many pts. of A.

- If A is an interval, then the set of all cluster pts of A is the closure of A.

Def^n: Given a fn $f: A \to \mathbb{R}$ and cluster pt. x_0 of A, we say that the limit of f at x_0 is equal to $L \in \mathbb{R}$ iff $\forall \varepsilon > 0$, there exists a positive constant $\delta = \delta(\varepsilon) > 0$ s.t.

$$|f(x) - L| < \varepsilon, \ \forall x \in A \text{ with } 0 < |x - x_0| < \delta.$$

We write $\lim_{x \to x_0} f(x) = L$.

eg: $\lim_{x \to 1} x^2 = 1$.

Given $\varepsilon > 0$ we need to find $\delta > 0$ s.t. $|x^2 - 1| < \varepsilon$, $\forall x$ with $|x-1| < \delta$.

Here $|x^2 - 1| = |x-1||x+1|$.

Since $0 < x < 2$ as $x \to 1$, we have $|x+1| < 3$.

$\therefore |x^2 - 1| < 3|x-1| \leq 3\delta = \varepsilon$.

Take $\delta = \varepsilon/3$ and the def^n is satisfied.

eg: $\lim_{x \to 2} \sqrt{x} = \sqrt{2}$.

Take $\delta = \sqrt{2}\varepsilon$. since $|\sqrt{x} - \sqrt{2}| = \dfrac{|x-2|}{|\sqrt{x} + \sqrt{2}|} \leq \dfrac{|x-2|}{\sqrt{2}}$.

eg: $\lim_{x \to 0} (2x + x^3) = 0$

<u>Sequential criterion of limits</u>: Let $f: A \to \mathbb{R}$ and x_0 is a cluster point. Then the following are equivalent:

(1) $\lim_{x \to x_0} f(x) = L$,

(2) For every seq^n (x_n) s.t. $x_n \in A$ and $x_n \to x_0$ and has the property $x_n \neq x_0$ for infinitely many $n \in \mathbb{N}$, then the seq^n $f(x_n) \to L$.

eg: $f(x) = \sin 1/x$ has no limit as $x \to 0$.

<u>Def^n</u>: Given $f: A \to \mathbb{R}$ and x_0 a cluster point of A, we say that $\lim_{x \to x_0} f(x) = +\infty$ if $\forall M \in \mathbb{R}$, $\exists$ $\delta = \delta(M) > 0$ s.t.

$$f(x) > M, \forall x \in A \text{ with } 0 < |x - x_0| < \delta.$$

Similarly, $\lim_{x \to x_0} f(x) = -\infty$, if $\forall$ $m \in \mathbb{R}$, $\exists$, $\delta = \delta(m) > 0$ s.t.

$$f(x) < m, \forall x \in A \text{ with } 0 < |x - x_0| < \delta.$$

eg: $\lim_{x \to 0} \frac{1}{|2x^2 + x|} = +\infty$. $\left[\frac{1}{|2x^2+x|} = \frac{1}{|x||2x+1|}. \right]$.

Since $x \to 0$, we assume $-1 < x < 1$ so, $-1 < 2x + 1 < 3$

$\Rightarrow |2x+1| < \max\{|-1|, |3|\}$
$= 3.$

$$\therefore \frac{1}{2x^2+x} > \frac{1}{3x} > \frac{1}{3\delta} = M.$$

Choose, $\delta = \min\{1, 1/3M\}$ suffices.

eg: $\lim_{x \to 0} \left(-\frac{1}{2x^2}\right) = -\infty$.

<u>Def^n</u>: Given a fn $f: A \to \mathbb{R}$ with A unbounded from above and $L \in \mathbb{R}$, then $\lim_{x \to +\infty} f(x) = L$ if for any $\varepsilon > 0$, $\exists$ $c = c(\varepsilon) \in \mathbb{R}$ s.t.

$$|f(x) - L| < \varepsilon, \forall x \in A \text{ with } x > c.$$

Similarly, $\lim_{x\to -\infty} f(x) = L$, if $\forall\ \varepsilon > 0$, $\exists\ c = c(\varepsilon) \in \mathbb{R}$ s.t.

$|f(x) - L| < \varepsilon$, $\forall x \in A$ with $x < c$.

eg: $\lim_{x\to +\infty} \frac{1}{x-3} = 0$.

Assume $x > 3$ so $x \neq 3$, given $\varepsilon > 0$, then if $x > c$ then,

$$\frac{1}{x-3} < \frac{1}{c-3} = \varepsilon.$$

Choose $c = 1/\varepsilon + 3$ //.

Theorem: Let $f: A \to \mathbb{R}$, and $g: B \to \mathbb{R}$ and let x_0 be a cluster point for $A \cap B$. If $\lim_{x\to x_0} f(x) = L$ and $\lim_{x\to x_0} g(x) = T$, then

(i) $\lim_{x\to x_0} (f(x) + g(x)) = L + T$,

(ii) $\lim_{x\to x_0} (f(x)\, g(x)) = LT$,

(iii) If $T \neq 0$, $\lim_{x\to x_0} \frac{f(x)}{g(x)} = \frac{L}{T}$.

Proof: Use sequential criterion and algebra of limits of seq^{n}s.

- $\lim_{x\to x_0} a x^n = a x_0^n$, $\forall a \in \mathbb{R}$
- $\lim_{x\to x_0} P(x) = P(x_0)$, $\forall P(x)$ polynomial.
- $\lim_{x\to x_0} \frac{P(x)}{Q(x)} = \frac{P(x_0)}{Q(x_0)}$, $\forall P(x), Q(x)$ polynomials with $Q(x_0) \neq 0$.

Squeeze/Sandwich Theorem: Let $A \subseteq \mathbb{R}$, and $x_0 \in \mathbb{R}$ a cluster pt. of A. Let $f, g, h: A \to \mathbb{R}$ s.t. $f(x) \leq g(x) \leq h(x)$ $\forall x \in A$, $x \neq x_0$ and $\lim_{x\to x_0} f(x) = L = \lim_{x\to x_0} h(x)$. Then the fn g admits a limit for $x \to x_0$ and $\lim_{x\to x_0} g(x) = L$.

eg: $\lim_{x \to 0} \frac{\sin x}{x} = 1$.

$$\cos x < \frac{\sin x}{x} < 1 \quad (\text{Prove this for } x \in (-\pi/2, \pi/2), x \neq 0).$$

Use Squeeze theorem now.

• The limit at a point does not always exist. This can happen if the fn takes different values from different sides as the fn approaches the cluster pt. eg Sign fuction.

<u>Def^n</u>: (i) If $x_0 \in \mathbb{R}$ is a cluster pt of $A \cap (x_0, +\infty)$, then we say that $L \in \mathbb{R}$ is the <u>right-hand limit of f at x_0</u> if $\forall\, \varepsilon > 0\ \exists\ \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(x) - L| < \varepsilon, \quad x \in A \text{ with } 0 < x - x_0 < \delta.$$

We write $\lim_{x \to x_0^+} f(x) = L$.

(ii) If $x_0 \in \mathbb{R}$ is a cluster point of $A \cap (-\infty, x_0)$, then we say that $L \in \mathbb{R}$ is the <u>left-hand limit of f at x_0</u> if $\forall\, \varepsilon > 0 .\ \exists\ \delta = \delta(\varepsilon) > 0$ s.t.

$$|f(x) - L| < \varepsilon, \quad x \in A \text{ with } -\delta < x - x_0 < 0.$$

We write $\lim_{x \to x_0^-} f(x) = L$.

<u>Thm</u>: If x_0 is a cluster point for the domain A, then

$$\lim_{x \to x_0} f(x) = L \text{ iff } \lim_{x \to x_0^-} f(x) = L = \lim_{x \to x_0^+} f(x).$$

eg: $\lim_{x \to 0} \frac{\sin x}{|x|}$ doesn't exist.

Recall, $\lim_{x \to 0} \frac{\sin x}{x} = 1$ (Squeeze Thm).

$$\Leftrightarrow \lim_{x \to 0^+} \frac{\sin x}{x} = 1 = \lim_{x \to 0^-} \frac{\sin x}{x}.$$

Now, $\lim_{x \to 0+} \frac{\sin x}{|x|} = \lim_{x \to 0+} \frac{\sin x}{x} = 1$.

But, $\lim_{x \to 0-} \frac{\sin x}{|x|} = \lim_{x \to 0-} \frac{\sin x}{-x} = -1$. //

Defⁿ: Given a ~~f~~ non-empty set $A \subset \mathbb{R}$, a fn $f: A \to \mathbb{R}$ and a pt. $x_0 \in A$ which is a cluster point for A, we say that f is continuous at the pt. x_0 iff $\lim_{x \to x_0} f(x) = f(x_0)$.

i.e. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$, $\forall x \in A$ and $|x - x_0| < \delta$.

Whenever $x_0 \in A$ is not a cluster point of A, we always say that the fn is continuous at that point.

Defⁿ: Given a fn $f: A \to \mathbb{R}$ we say $f(x)$ is continuous if $f(x)$ is continuous $\forall x \in A$.

- We now assume any pt. in the domain is a cluster pt.

eg: $f(x) = 1/x$ is continuous on $A = (-\infty, 0) \cup (0, +\infty)$.

$$g(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$$ is discontinuous at the pt. 0.

- The power fn. x^n is continuous $\forall n \in \mathbb{N}$.
  (Use induction & algebra of limits.)

- Any poly is a continuous fn.

Examples of discontinuous functions:

(1) $f(x) = \text{sign}|x| = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0 \end{cases}$. Here $\lim_{x \to 0} f(x) = 1$ but $f(0) = 0$.

(2) $f(x) = \text{sign}\, x = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$ RHL & LHL are different.

(3) $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ RHL & L.H~~S~~ ~~as~~ $x \to 0$ are $+\infty$ & $-\infty$.

(4) $f(x) = \begin{cases} \sin 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ discontinuous at $x_0 = 0$ since RHL as $x \to 0$ doesn't exist.

<u>Algebra of continuous functions</u>: Given a fn $f, g : A \to \mathbb{R}$ and a pt. $x_0 \in A$,

1. If $f(x)$ is cont. at x_0, then so is $g(x) = f(x) + c$ for any $c \in \mathbb{R}$.
2. If $f(x)$ & $g(x)$ are cont. at x_0 then so is $h(x) = g(x) + f(x)$ and $w(x) = f(x)\,g(x)$.
3. If $f(x)$ & $g(x)$ are cont. at x_0 and $g(x) \neq 0$ then $h(x) = \frac{f(x)}{g(x)}$ is cont. at x_0.

<u>Continuity of composition fn</u>: Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ be two fns with $f(A) \subset B$ and consider a point $y_0 \in B$ s.t. $y_0 = f(x_0)$ for some $x_0 \in A$. If f is cont. at x_0 and g at y_0, then $h := g \circ f$ is cont. at x_0.

<u>Continuity of the inverse fn</u>: Given a bijective fn $f : A \to B$. If f is cont. on A, then the inverse fn $f^{-1} : B \to A$ is cont. on B.

<u>Intermediate Value Theorem</u>: Let $f : [a, b] \to \mathbb{R}$ be cont, then for any value c between $f(a)$ & $f(b)$, $\exists\ x_0 \in [a, b]$ s.t. $f(x_0) = c$.

- IVT is false if f is not cont. <u>eg</u>: $f : [-1, 1] \to \mathbb{R}$ with $f(x) = 1\ \forall\ 0 < x \leq 1$ and $f(x) = -1\ \forall\ -1 \leq x \leq 0$.

Corollary: Any polynomial with odd degree has at least one real root.

OR, Let $P: \mathbb{R} \to \mathbb{R}$ be defined by $P(x) = a_{2n+1} x^{2n+1} + \dots + a_1 x + a_0$, where $n \in \mathbb{N}$ and $a_i \in \mathbb{R}$ with $a_{2n+1} \neq 0$. Then there is at least one pt $x_0 \in \mathbb{R}$ s.t. $P(x_0) = 0$.

Proof: Let $P(x)$ be a real polynomial with odd degree. WLOG, we assume $a_{2n+1} > 0$, then $\lim_{x \to -\infty} P(x) = -\infty$ and $\lim_{x \to +\infty} P(x) = +\infty$.

By def^{n} of limits, this implies, $\exists$ at least two pts x_1 and x_2 s.t. $P(x_1) > 0$ and $P(x_2) < 0$. By the IVT we now get an x_0, s.t. $x_2 < x_0 < x_1$ with $P(x_0) = 0$.

Similarly the case for $a_{2n+1} < 0$. //.

Weierstrass Extreme Value Theorem: Let $f: [a, b] \to \mathbb{R}$ be cont. fn defined on $[a, b]$, a closed and bounded interval, then the fn admits a max^{m} and min^{m} on $[a, b]$.

Corollary: Let $[a, b]$ be closed, bounded interval, and let $f: [a, b] \to \mathbb{R}$ be cont. Then f is bounded on $[a, b]$.

- Closed ~~Bounded~~ is essential: $f(x) = x$ on $[0, \infty)$.
- Bounded is essential: $f(x) = 1/x$ on $(0, 1]$ is not bounded above.
- Continuity is essential: $f(x) = \begin{cases} 1 - x, & \text{on } (0, 1] \\ 0, & \text{when } x = 0. \end{cases}$