<u>Convergence of a Series:</u> Let (b_n) be a seq^n. An infinite series is a formal exp^n of the form $b_1 + b_2 + b_3 + \cdots$.

The corresponding sequence of partial sums (s_m) is defined by

$$s_m = b_1 + b_2 + \cdots + b_m .$$

We say that the series $\sum_{i=0}^{\infty} b_i$ converges to B if the seq^n (s_m) converges to B. We write then, $\sum_{i=0}^{\infty} b_i = B$.

~~e.g.: Consider~~ $\cancel{\sum_{n=1}^{\infty} \frac{1}{n^2}}$.

<u>Def^n:</u> A seq^n (a_n) is <u>increasing</u> if $a_n \leq a_{n+1}$ $\forall\, n \in \mathbb{N}$ and <u>decreasing</u> if $a_n \geq a_{n+1}$ $\forall\, n \in \mathbb{N}$. A seq^n is <u>monotone</u> if it is either increasing or decreasing.

<u>Monotone Convergence Theorem:</u> If a seq^n is monotone and bounded, then it converges.

<u>eg</u>: Consider $\sum_{n=1}^{\infty} \frac{1}{n^2}$.

Here $$s_m = 1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{m^2} < 1 + \frac{1}{2\cdot 1} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3} + \cdots + \frac{1}{m(m-1)}$$
$$= 1 + \left(1 - \tfrac{1}{2}\right) + \left(\tfrac{1}{2} - \tfrac{1}{3}\right) + \cdots + \left(\tfrac{1}{m-1} - \tfrac{1}{m}\right)$$
$$= 1 + 1 - \tfrac{1}{m}$$
$$< 2.$$

By the MCT, $\sum_{n=1}^{\infty} 1/n^2$ converges to some limit.

<u>eg</u>: <u>Harmonic Series</u>: $\sum_{n=1}^{\infty} 1/n$ diverges.

Here, $s_m = 1 + 1/2 + \cdots + 1/m$. Clearly, $s_4 = 1 + 1/2 + (1/3 + 1/4)$
$$\geq 1 + 1/2 + (1/4 + 1/4) = 2.$$

Similarly, $$s_{2^k} = 1 + 1/2 + (1/3 + 1/4) + (1/5 + \cdots + 1/8) + \cdots + \left(\tfrac{1}{2^{k-1}+1} + \cdots + \tfrac{1}{2^k}\right)$$
$$\geq 1 + 1/2 + (1/4 + 1/4) + \cdots + \left(1/2^k + 1/2^k + \cdots + 1/2^k\right)$$
$$= 1 + k/2$$ which is unbounded.

<u>Cauchy Condensation Test</u>: Suppose (b_n) is decreasing and $b_n \geq 0$ $\forall n \in \mathbb{N}$. Then, the series $\sum_{n=1}^{\infty} b_n$ converges iff the series $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges.

<u>Proof:</u> Assume that $\sum_{n=0}^{\infty} 2^n b_{2^n}$ converges. Since every convergent seq^n is bounded so, the partial sums $t_k = b_1 + 2b_2 + \dots + 2^k b_{2^k}$ are bounded. That is, $\exists$ $M > 0$ s.t. $t_k \leq M$ $\forall k \in \mathbb{N}$.

Since $b_n \geq 0$, so the partial sums are increasing. To show that $\sum_{n=1}^{\infty} b_n$ converges we need to show that $s_m = b_1 + \dots + b_m$ is bounded.

We fix m and let k be s.t. $m \leq 2^{k+1} - 1$. Then,

$$s_m \leq s_{2^{k+1}-1} = b_1 + (b_2 + b_3) + (b_4 + \dots + b_7) + \dots + (b_{2^k} + \dots + b_{2^{k+1}-1})$$

$$\leq b_1 + (b_2 + b_2) + \dots + (b_{2^k} + \dots + b_{2^k})$$

$$= b_1 + 2b_2 + \dots + 2^k b_{2^k} = t_k.$$

Thus, $s_m \leq t_k \leq M$ so, (s_m) is bounded and by MCT we conclude $\sum_{n=1}^{\infty} b_n$ converges.

The other part: $\sum_{n=0}^{\infty} 2^n b_{2^n}$ diverges $\Rightarrow \sum_{n=1}^{\infty} b_n$ diverges is H/ω.

<u>Def^n</u>: Let (a_n) be a seq^n of real nos, let $n_1 < n_2 < \dots$ be an increasing seq^n of natural nos. Then the seq^n $(a_{n_1}, a_{n_2}, \dots)$ is called a <u>subseq^n</u> of (a_n), denoted by (a_{n_k}).

<u>eg</u>: $(a_n) = 1/n$, then $(1/2, 1/4, 1/8, \dots)$ is a subseq^n.

$(1, 1, 1/3, 1/3, \dots)$ is <u>NOT</u> a subseq^n.

<u>Theorem:</u> Subsequences of a convergent seq^n converge to the same limit as the original seq^n.

<u>Proof:</u> Let $(a_n) \to a$, let (a_{n_k}) be a subseq^n.

Given $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$ s.t. $|a_n - a| < \varepsilon$ whenever $n \geq N$.

Since $n_k \geq k$ $\forall k$, $|a_{n_k} - a| < \varepsilon$ whenever $k \geq N$. //.

<u>Eg:</u> $(1, -1/2, 1/3, -1/4, 1/5, -1/5, 1/5, -1/5, \ldots)$ is divergent since the subseq^{n}s $(1/5, 1/5, \ldots)$ and $(-1/5, -1/5, \ldots)$ have different limits.

<u>Bolzano-Weierstrass Theorem:</u> Every bounded seq^n contains a convergent subseq^n.

<u>Proof:</u> Let (a_n) be a bounded seq^n such that $\exists$ $M > 0$ satisfying $|a_n| \leq M$ $\forall$ $n \in \mathbb{N}$.

Bisect the interval $[-M, M]$ into two: $[-M, 0]$ and $[0, M]$.

At least one of this interval contains an infinite no. of the terms in the seq^n (a_n), say I_1.

Let $a_{n_1} \in I_1$.

Next we bisect I_1 into closed intervals of equal length and pick one interval, say I_2 which has an infinite no. of terms from (a_n) to choose from. Let $a_{n_2} \in I_2$.

We keep on constructing $I_3, I_4, \ldots, I_k$ so that $n_k > n_{k-1} > \cdots > n_2 > n_1$ and $a_{n_k} \in I_k$.

We also have $I_1 \supseteq I_2 \supseteq \ldots$, so there exists at least one point $x \in \mathbb{R}$ that is contained in every I_k. We claim that $(a_{n_k}) \to x$.

Let $\varepsilon > 0$. By construction, the <u>length</u> of I_k is $M\left(\frac{1}{2}\right)^{k-1}$ and this converges to 0. Choose N s.t. $k \geq N$ so, the length of $I_k < \varepsilon$. Since $a_{n_k}, x \in I_k$ we have. $|a_{n_k} - x| < \varepsilon$. //.

<u>Def^n</u>: A seq$^{\underline{n}}$ (a_n) is called a Cauchy seq$^{\underline{n}}$ if for every $\varepsilon > 0$, there exists an $N \in \mathbb{N}$ s.t. whenever $m, n \geq N$, we have $|a_n - a_m| < \varepsilon$.

<u>Theorem:</u> Every convergent seq$^{\underline{n}}$ is a Cauchy seq$^{\underline{n}}$.

<u>Proof</u>: Let $(x_n) \to x$. To show $|x_n - x_m| < \varepsilon$ for suitable m, n just apply triangle inequality //

<u>Theorem</u>: Cauchy seq$^{\underline{n}}$s are bounded.

<u>Proof</u>: Given $\varepsilon = 1$, $\exists$ N s.t. $|x_n - x_m| < 1$ $\forall$ $m, n \geq N$.

Thus, we must have, $|x_n| < |x_N| + 1$ $\forall$ $n \geq N$.

Then, $M = \max\{|x_1|, |x_2|, \ldots, |x_{N-1}|, |x_N| + 1\}$ is a bound for the seq$^{\underline{n}}$ (x_n).

<u>Cauchy Criterion</u>: A seq$^{\underline{n}}$ converges iff it is a Cauchy seq$^{\underline{n}}$.

<u>Proof</u>: $\Rightarrow$ proved above.

$\Leftarrow$ They Cauchy seq$^{\underline{n}}$ (x_n) is bounded. So, by B-W Theorem, we have a convergent subseq$^{\underline{n}}$ (x_{n_k}). Let $x_{n_k} \to x$.

Let $\varepsilon > 0$, since (x_n) is Cauchy, so $\exists$ N s.t. $|x_n - x_m| < \varepsilon/2$; whenever $n, m \geq N$. Again, choose a term in the subseq$^{\underline{n}}$ (x_{n_k}), say x_{n_ℓ}, with $n_\ell \geq N$ and, $|x_{n_\ell} - x| < \varepsilon/2$.

For $n \geq N$ we have,
$$|x_n - x| = |x_n - x_{n_\ell} + x_{n_\ell} - x|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon. \quad //$$