Lecture 10 : Determinants.

One viewpoint: The determinant provides an explicit "formula" for each entry of A^{-1} and $A^{-1}b$ (recall sol^n. of linear eq^{n}s.).

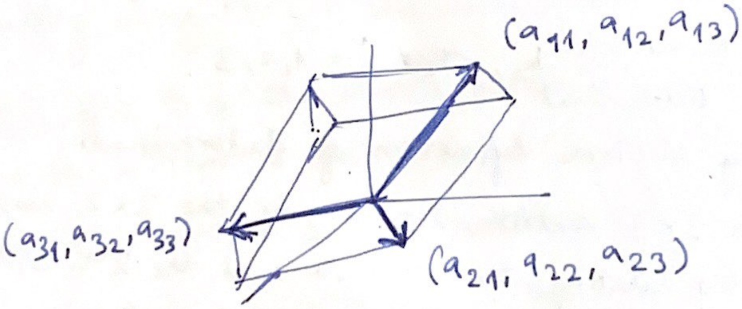
Uses for determinants:

(1) Test for invertibility: If $\det(A) = 0$ then A is singular.
If $\det(A) \neq 0$ then A is invertible.

(In the next chapter we want to understand $\det(A - \lambda I)$ better which is a polynomial of degree n in λ and has n roots.)

(2) Geometry: The $\det(A)$ is the volume of a box in an n-dimensional space. The edges are the ~~columns of A~~ rows. The columns give a different box with the same volume.

eg:
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$



volume $= \det(A)$

(3) Formula for each pivot: determinant $= \pm$ (product of the pivots).

(4) Independence/Dependence of $A^{-1}b$ on each element of b: If we change just one parameter ~~then~~ in an expt/eq. then the ~~coef~~ coefficients in A^{-1} is a ratio of determinants.

- There are more than one way to define a determinats.
- What matters for us are the algebraic properties that a determinant posess.

The three most basic properties are:

(a) $\det I = 1$ (think of an unit cube)

(b) Exchanging a row reverses the sign

(c) It is linear in each row separately.

<u>Properties of the Determinant:</u>

Consider the system
$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned}\right\} \Rightarrow x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}$$

$$x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}$$

Common ~~numerator~~ denominator is then defined as the determinant of the co-eff. matrix.

Similarly for a 3×3 system $Ax = b$, we again get sol^{n}s

$$x_i = \frac{\text{Numerator } i}{\text{Determinat}}, \quad i = 1, 2, 3.$$

There is actually another definition of determinants:

<u>Def^n:</u> Let A be a 3×3 matrix, let A_{jk} be the 2×2 matrix obtained from A by deleting the j^{th} ~~col^m~~ row and the k^{th} col^m.

The <u>cofactor</u> of a_{jk} is then defined as $C_{jk} = (-1)^{j+k} \det A_{jk}$.

The <u>determinant</u> is then $\det A = \underbrace{a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}}_{\text{expansion along the 1st row}}$.

- We can extend this def^n immediately.

<u>Thm:</u> Let A be an $n\times n$ matrix, then $\det A$ may be obtained by cofactor exp^n along any row/col^m of A:

$$\det A = a_{j1}C_{j1} + a_{j2}C_{j2} + \dots + a_{jn}C_{jn}.$$

<u>Corollary</u>: If A has a 0 row/col^{m} then $\det A = 0$.

<u>Corollary</u>: For any sq. matrix A, $\det A = \det A^T$.

<u>Theorem</u>: The det. of a LT/UT matrix is the product of its diagonal entries.

<u>Theorem</u>: $\det I_n = 1$.

<u>Theorem</u>: The determinant changes sign when two rows are reversed.

<u>Corollary</u>: If two rows of A are equal then $\det A = 0$.

<u>Theorem</u>: Let ~~A~~ B be the matrix obtained by multiplying a row of A by β. Then $\det B = \beta \det A$.

<u>Proof</u>: Let the j^{th} row of B ~~be~~ is obtained from A by multiplying with β.

We have, $B_{jk} = A_{jk}$, $k = 1, 2, \dots, n$.

↳ matrices obtained by deleting j^{th} row, k^{th} col.

In particular the (j,k) cofactors of A and B are equal. Expanding along the j^{th} row now gives the result. //

<u>Theorem</u>: Let B be the matrix obtained from A by adding β times the k^{th} row to the j^{th} row. Then $\det B = \det A$.

<u>Proof</u>: For any matrix A and row vector $r = (r_1\ r_2 \dots r_n)$,

$r_1 C_{j1} + r_2 C_{j2} + \dots + r_n C_{jn}$ is the det. of the matrix obtained from A by replacing the j^{th} row with r.

The j^{th} row of B is $b_j = a_j + \beta a_k$. Now, expanding along the j^{th} row, $\det B = (a_j + \beta a_k) \cdot c_j^T$ where $c_j = (c_{j1}\ c_{j2} \dots c_{jn})$.

$$= a_j \cdot c_j^T + \beta \underbrace{(a_k \cdot c_j^T)}_{=0}$$

$$= \det A$$

since here the k^{th} row and the j^{th} row are equal. for $k \neq j$. //

Ex: Suppose A is a 4×4 matrix, with $\det A = 11$. Let a_1, a_2, a_3, a_4 be the rows of A. If B is obtained from A by replacing a_3 by $3a_1 + 7a_3$. What is $\det B$?

Ans: $\det B = (3a_1 + 7a_3) \cdot c_3^T = 3a_1 \cdot c_3^T + 7a_3 \cdot c_3^T = 7(a_3 \cdot c_3^T) = 77$.//

Theorem: A sq. matrix A is invertible iff $\det A = 0$.

Proof: Starting with A, we perform elementary row operations to get a matrix in the row echelon form and thus triangular:

$$A \sim A_1 \sim A_2 \sim \dots \sim A_k.$$

Here if $\det A_{i-1} \neq 0$ then $\det A_i \neq 0$.

In particular, $\det A = 0$ iff $\det A_p = 0$.

If all diagonal entries of A_p are non-zero then $\det A_p \neq 0$.

$\hookrightarrow$ In this case A_p is invertible since we have n pivots in A_p.

If at least one diagonal entry of A_p is 0 then $\det A_p = 0$ and in this case A_p is not invertible because we have $< n$ pivots. //

Theorem: Let $B = \beta A$, then $\det B = \beta^n \det A$.

Proof: Use induction.

Theorem: $\det AB = \det A \cdot \det B$.

Corollary: For any sq-matrix, $\det(A^k) = (\det A)^k$.

Corollary: If A is invertible, then $\det(A^{-1}) = \dfrac{1}{\det A}$.

Proof of Theorem: Let $d(A) = \dfrac{\det AB}{\det B}$. We show that $d(A)$ has the following properties: (i) $d(I) = 1$.

(ii) $d(A)$ changes sign when two rows are exchanged.

(iii) $d(A)$ depends linearly on the 1st row.

Then $d(A)$ must equal $\det(A)$.

(i) is easy to verify.

(ii) If two rows of A are exchanged then two rows of AB are also exchanged.

Thus the sign of d changes since sign of $\det AB$ changes.

(iii) A linear combination in the 1st row of A gives the same linear combination in the 1st row of AB. //.

<u>Formula for the inverse of a matrix:</u>

Recall, $A \in \mathbb{R}^{n\times n}$, $\det A = a_{j1}C_{j1} + a_{j2}C_{j2} + \cdots + a_{jn}C_{jn}$.

then $C_{jk} = (-1)^{j+k}\det A_{jk}$ is the (j,k)-cofactor of A and,

$a_j = (a_{j1}\ a_{j2}\ \cdots\ a_{jn})$ is the j^{th} row of A.

If $c_j = (c_{j1}\ c_{j2}\ \cdots\ c_{jn})$ then, $\det A = a_j \cdot c_j^T$.

In fact, $a_j \cdot c_k^T = \begin{cases} \det A & \text{if } j = k \\ 0 & \text{if } j \neq k \end{cases}$ (as this gives det of a matrix with two rows equal).

We now form the <u>cofactor matrix</u> $\mathrm{Cof}(A)$ as,

$$\mathrm{Cof}(A) = \begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1n} \\ c_{21} & c_{22} & \cdots & c_{2n} \\ \vdots & & & \\ c_{n1} & c_{n2} & \cdots & c_{nn} \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}.$$

Then, $$A(\mathrm{Cof}(A))^T = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix} (c_1^T\ c_2^T\ \cdots\ c_n^T)$$

$$= \begin{pmatrix} a_1c_1^T & a_1c_2^T & \cdots & a_1c_n^T \\ a_2c_1^T & a_2c_2^T & \cdots & a_2c_n^T \\ \vdots & & & \\ a_nc_1^T & a_nc_2^T & \cdots & a_nc_n^T \end{pmatrix}$$

$$= \begin{pmatrix} \det A & & & O \\ & \det A & & \\ & & \ddots & \\ O & & & \det A \end{pmatrix} = \det A \cdot I_n.$$

So, we have obtained,

$$A(\mathrm{Cof}(A))^T = \det(A)\, I_n .$$

$$\Rightarrow A\left(\frac{1}{\det A}\right)(\mathrm{Cof}(A))^T = I_n \quad \text{if } \det A \neq 0 .$$

$$\Rightarrow A^{-1} = \frac{1}{\det A}(\mathrm{Cof}(A))^T . \quad \text{if } \det A \neq 0 .$$

(This is very computationally intensive!!)

eg: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\mathrm{Cof}(A) = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} = \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$.

$$\therefore A^{-1} = \frac{1}{\det A}(\mathrm{Cof}(A))^T = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} . //$$

Q. When does an integer matrix have an integer inverse?

↳ every entry of the matrix is an integer.

Theorem: An invertible integer matrix $A \in \mathbb{R}^{n\times n}$ has an integer inverse A^{-1} iff $\det A = \pm 1$.

Proof: Suppose $A \in \mathbb{R}^{n\times n}$ is an invertible integer matrix. So $\det A \neq 0$,. and ~~det~~ $\det A \in \mathbb{Z}$, $(\mathrm{Cof}(A))^T$ is an integer matrix.

If A^{-1} is also an integer matrix then $\det(A^{-1}) \in \mathbb{Z}$.

Now, $\det(A)\det(A^{-1}) = \det(AA^{-1}) = 1$. $\Rightarrow \det A = \pm 1$.

Let on the other hand, $\det A = \pm 1$ then $A^{-1} = \pm(\mathrm{Cof}(A))^T$. //.

* We can generate integer matrices with an integer inverse.
  - Start with U_0 an upper triangular matrix with integer entries and diagonal entries either 1 or -1. Then $\det U_0 = \pm 1$.
  - Perform any seq^n of elementary row operations .. (interchange two rows or ~~multiply a row~~ add a multiple of one row to another)
  - This generates a seq^n of matrices with integer entries and det $= \pm 1$.

$$U_0 \sim U_1 \sim \dots \sim U_k .$$

Cramer's Rule:

Recall, if A is invertible then a sol$^{\underline{n}}$ of $Ax=b$ is $x = A^{-1}b$.

Thus,
$$x = \frac{1}{\det A}\begin{pmatrix} C_{11} & C_{21} & \cdots & C_{n1} \\ C_{12} & C_{22} & \cdots & C_{n2} \\ \vdots & & & \\ C_{1n} & C_{2n} & \cdots & C_{nn} \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

This gives us,
$$x_1 = \frac{1}{\det A}\underbrace{(b_1C_{11} + b_2C_{21} + \cdots + b_nC_{n1})}_{\shortparallel}$$
$$\det\begin{pmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \\ b_n & a_{n2} & \cdots & a_{nn} \end{pmatrix}$$

Similarly,
$$x_2 = \frac{1}{\det A}\underbrace{(b_1C_{12} + b_2C_{22} + \cdots + b_nC_{n2})}_{\shortparallel}$$
$$\det\begin{pmatrix} a_{11} & b_1 & a_{13} & \cdots & a_{1n} \\ a_{21} & b_2 & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \\ a_{n1} & b_n & a_{n3} & \cdots & a_{nn} \end{pmatrix}.$$

In general we get the following:

Cramer's Rule: Let $A \in \mathbb{R}^{n\times n}$ be an invertible matrix. Let $b \in \mathbb{R}^n$ and let A_i be the matrix obtained from A by replacing the ith column ~~with of A~~ by b. Then the sol$^{\underline{n}}$ to $Ax=b$ is,
$$x = \frac{1}{\det A}\begin{pmatrix} \det A_1 \\ \det A_2 \\ \vdots \\ \det A_n \end{pmatrix}.$$

(This is computationally very intensive!!)

Another formula for determinants:

~~The~~ $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$

$$\det\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{array}{l} +a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} \\ -a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31} \end{array}$$

We want to derive these formulas from the defining properties of the det : (i) $\det I_n = 1$.

(ii) Exchange rows changes sign.

(iii) Linear in the 1^{st} row.

Let's look at 2×2 case, here $(a\ b) = (a\ 0) + (0\ b)$

$(c\ d) = (c\ 0) + (0\ d)$.

By (iii) $\det\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \det\begin{pmatrix} a & 0 \\ c & d \end{pmatrix} + \det\begin{pmatrix} 0 & b \\ c & d \end{pmatrix}$

$$= \det\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} + \det\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + \det\begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} + \det\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix}$$

- Each row splits into n co-ordinate directions.
- So, the exp^n has n^n terms.
- Most of the terms equal 0.
- When two rows point in the same co-ordinate direction, one is a multiple of another. So, $\det\begin{pmatrix} a & 0 \\ c & 0 \end{pmatrix} = \det\begin{pmatrix} 0 & b \\ 0 & d \end{pmatrix} = 0$.
- We pay attention only when rows point in different dir$^{\underline{n}}$.

The non-zero terms have to come in different columns. That is, they are a reordering or a permutation of $1, 2, \dots, n$. Thus they produce $n!$ determinants.

eg. For $n = 3$ case we have the following:

$$\det \underbrace{\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}}_{A} = \det \begin{pmatrix} a_{11} & & \\ & a_{22} & \\ & & a_{33} \end{pmatrix} + \det \begin{pmatrix} & a_{12} & \\ & & a_{23} \\ a_{31} & & \end{pmatrix}$$

$$+ \det \begin{pmatrix} & & a_{13} \\ a_{21} & & \\ & a_{32} & \end{pmatrix} + \det \begin{pmatrix} a_{11} & & \\ & & a_{23} \\ & a_{32} & \end{pmatrix} + \det \begin{pmatrix} & a_{12} & \\ a_{21} & & \\ & & a_{33} \end{pmatrix}$$

$$+ \det \begin{pmatrix} & & a_{13} \\ & a_{22} & \\ a_{31} & & \end{pmatrix}.$$

Only these are non-zero, otherwise a column is repeated and hence they are det. zero as one col^{m} becomes 0.

$$\text{So, } \det A = a_{11}a_{22}a_{33} \det P_1 + a_{12}a_{23}a_{31} \det P_2 + a_{13}a_{21}a_{32} \det P_3$$

$$+ a_{11}a_{23}a_{32} \det P_4 + a_{12}a_{21}a_{33} \det P_5 + a_{13}a_{22}a_{31} \det P_6 .$$

This suggests the following formula for $A \in \mathbb{R}^{n \times n}$:

$$\det A = \sum_{\text{all } P\text{'s}} (a_{1\alpha} a_{2\beta} \cdots a_{n\nu}) \det P$$

where P is a permutation matrix ~~and~~.