Similarity of Matrices and Diagonalization of Matrices

Def^n: Let A and B be $n \times n$ matrices. We say that A is similar to B, denoted by $A \sim B$ sometimes, if there exists an invertible matrix P such that, $A = PBP^{-1}$.

- $A \sim B \Rightarrow B \sim A$ (just multiply by P^{-1} on left and P on right.)

Theorem: If $A \sim B$, then the following are true:

(i) $\text{rank}(A) = \text{rank}(B)$. (Easy.)

(ii) $\det(A) = \det(B)$ (Easy.)

(iii) A and B have the same eigenvalues.

Proof of (iii): $A = PBP^{-1}$ for some matrix P.

$$\det(A - \lambda I) = \det(A - \lambda PP^{-1}) = \det(PBP^{-1} - \lambda PP^{-1})$$
$$= \det(P(B - \lambda I)P^{-1}) = \det P \det(B - \lambda I)\det(P^{-1}).$$

So, A and B have the same eigenvalues as the $p(\lambda)$ is same.//

Recall, if A is a triangular matrix, then the eigenvalues of A are its diagonal entries.

Def^n: A matrix A is called diagonalizable if it is similar to a diagonal matrix D. i.e. there exist an invertible matrix P such that $A = PDP^{-1}$.

Theorem: A matrix A is diagonalizable iff there is a basis $\{v_1, v_2, \dots, v_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of A.

Proof: Let A be diagonalizable. Let $P = (v_1\ v_2 \dots v_n)$ and

$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots & \lambda_n \end{pmatrix}$ such that $A = PDP^{-1}$. Multiplying by P we get,

$$AP = PD$$

i.e. $(Av_1\ Av_2 \dots Av_n) = (\lambda_1 v_1\ \lambda_2 v_2 \dots \lambda_n v_n)$.

$\Rightarrow Av_i = \lambda_i v_i$.

So, the col^m of P, $v_1, v_2, \dots, v_n$ are the eigenvectors of A and they form a basis for $\mathbb{R}^n$ since P is invertible.

Conversely, let $\{v_1, v_2, \dots, v_n\}$ be a basis of $\mathbb{R}^n$ where v_i's are the eigenvectors of A. Let λ_i be the eigenvalue associated to v_i. Set $P = (v_1\ v_2\ \cdots\ v_n)$ which is now invertible and let

$D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. It is now easy to see that $PD = AP$

$\Rightarrow A = PDP^{-1}$. //.

* The problem of diagonalization of a matrix A is equivalent to finding a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.
* It is not always possible to diagonalize a matrix.

<u>Theorem</u>: Let $A \in \mathbb{R}^{n\times n}$ has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then A is diagonalizable.

<u>Proof</u>: Each eigenvalue λ_i gives an eigenvector v_i. The set $\{v_1, v_2, \dots, v_n\}$ is linearly indep. because they correspond to distinct eigenvalues. So, $\{v_1, v_2, \dots, v_n\}$ is a basis of $\mathbb{R}^n$ and hence A is diagonalizable. //.

* What if A has repeated eigenvalues?

<u>Theorem</u>: A matrix A is diagonalizable iff the algebraic and geometric multiplicities of each eigenvalue are equal.

<u>Proof</u>: Let the distinct eigenvalues be $\lambda_1, \lambda_2, \dots, \lambda_p$ and let the alg. and geom. multiplicities be $k_1, k_2, \dots, k_p$ and $g_1, g_2, \dots, g_p$ resp.

Let $k_i = g_i$. Since $k_1 + k_2 + \dots + k_p = n$ so, $g_1 + g_2 + \dots + g_p = n$. That is, there exist n linearly indep. eigen. vectors of A so A is diagonalizable.

Conversely, since A has n linearly indep. eigenvectors and $g_i \le k_i$ & $\sum_{i=1}^{p} k_i = n$ so the only possibility is $g_i = k_i$ so that $\sum_{i=1}^{n} g_i = n$. //.

<u>Ex!</u> Determine if $A = \begin{pmatrix} 2 & 4 & 3 \\ -4 & -6 & -3 \\ 3 & 3 & 1 \end{pmatrix}$ is diagonalizable. If yes, find a matrix P that diagonalizes A.

Sol^n: $P(\lambda) = \det(A - \lambda I) = -(\lambda - 1)(\lambda + 2)^2$

Eigenvalues are $\lambda_1 = 1$, $\lambda_2 = -2$.

For $\lambda_2 = -2$, we get $A - \lambda_2 I \xrightarrow{\text{rref}} \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$. Here $g_2 = 1 \leq k_2 = 2$.

So, by the previous results, A is not diagonalizable. //.

<u>Remark:</u> The matrix P is not unique. Multiplying an eigenvector by a constant gives a different P.

<u>Some other important properties:</u>

1. trace of a matrix = sum of its eigenvalues.
2. determinant of a matrix = product of its eigen values.
3. The eigenvalues of A^2 are exactly $\lambda_1^2, \lambda_2^2, \dots, \lambda_n^2$ and every eigenvector of A is also an eigenvector of A^2. (Just multiply $Ax = \lambda x$ by A)
4. The above generalizes to higher powers of A as well.
5. In general the eigen values of AB are not the product of eigenvalues of A and B. eg. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $AB = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

eigenvals: 0, 0, 1

<u>Theorem:</u> Diagonalizable matrices share the same P iff $AB = BA$.

<u>Proof:</u> If $A = PD_1P^{-1}$, $B = PD_2P^{-1}$. Then $AB = PD_1P^{-1}PD_2P^{-1}$

$= PD_1D_2P^{-1}$

& $BA = PD_2D_1P^{-1}$.

But $D_1D_2 = D_2D_1$ (Since diagonal matrices always commute).

Conversely, if $AB = BA$, then, let $Ax = \lambda x \Rightarrow ABx = BAx = B\lambda x = \lambda Bx$

So, x and Bx are both eigen vectors of A, sharing the same λ. i.e. if we assume the eigenvectors are distinct (wlog) then the eigenspace is 1-dimensional so Bx is a multiple of x and hence x is an eigenvector of B as well as A. //.

For symmetric matrices we can even prove the following:

Theorem: Let A be a symm. matrix. If v_1 and v_2 are eigenvectors of A corresponding to the distinct eigenvalues λ_1 and λ_2, then $v_1 \perp v_2$ i.e. $v_1^T \cdot v_2 = 0$.

Proof: We have, $\lambda_1 v_1^T v_2 = (\lambda_1 v_1)^T v_2 = (Av_1)^T v_2 = v_1^T A^T v_2$

$$= v_1^T A v_2 \text{ (since } A^T = A) = v_1^T (\lambda_2 v_2) = \lambda_2 v_1^T v_2.$$

$$\Rightarrow (\lambda_1 - \lambda_2) v_1^T v_2 = 0 \Rightarrow v_1^T v_2 = 0 \text{ (since } \lambda_1 \neq \lambda_2) \text{ //.}$$

Theorem: Any symmetric matrix is diagonalizable. In fact, it gives an orthonormal basis of $\mathbb{R}^n$ of eigenvectors of the matrix.
