Theorems for Differentiable Functions

Dr Manjil P. Saikia

IIIT Manipur

January 18, 2023

Rolle's Theorem

Rolle's Theorem

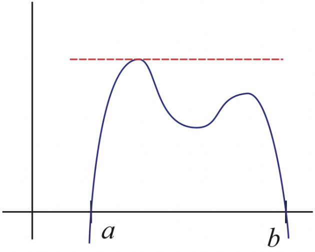
Theorem (Rolle's Theorem)

Let $f : [a, b] \to \mathbb{R}$ *be a continous function on a closed interval and it is differentiable at any point in the open interval* (a, b)*. Moreover assume that* $f(a) = f(b) = 0$*. Then there exists at least one point* $c \in (a, b)$ *such that* $f'(c) = 0$*.*

Rolle's Theorem

Theorem (Rolle's Theorem)

Let $f : [a, b] \to \mathbb{R}$ *be a continous function on a closed interval and it is differentiable at any point in the open interval* (a, b)*. Moreover assume that* $f(a) = f(b) = 0$*. Then there exists at least one point* $c \in (a, b)$ *such that* $f'(c) = 0$*.*



Proof of Rolle's Theorem

Proof of Rolle's Theorem

- Since $f(x)$ is continous, so by the Weirstraß Extreme Value Theorem the function admits a maximum and a minimum.

Proof of Rolle's Theorem

- Since $f(x)$ is continous, so by the Weirstraß Extreme Value Theorem the function admits a maximum and a minimum.
- There can be two cases:

Proof of Rolle's Theorem

- Since $f(x)$ is continous, so by the Weirstraß Extreme Value Theorem the function admits a maximum and a minimum.
- There can be two cases:
  - Case 1: Both the maximum and minimum are attained at the extrema of the interval.

Proof of Rolle's Theorem

- Since $f(x)$ is continous, so by the Weirstraß Extreme Value Theorem the function admits a maximum and a minimum.
- There can be two cases:
  - Case 1: Both the maximum and minimum are attained at the extrema of the interval.
  - Case 2: At least one of the maximum or minimum is attained at an internal point.

Proof of Rolle's Theorem

- Since $f(x)$ is continous, so by the Weirstraß Extreme Value Theorem the function admits a maximum and a minimum.
- There can be two cases:
  - Case 1: Both the maximum and minimum are attained at the extrema of the interval.
  - Case 2: At least one of the maximum or minimum is attained at an internal point.
- Case 1: Here $\max_{(a,b)} f(x) = \min_{(a,b)} f(x) = 0$,

Proof of Rolle's Theorem

- Since $f(x)$ is continous, so by the Weirstraß Extreme Value Theorem the function admits a maximum and a minimum.
- There can be two cases:
  - Case 1: Both the maximum and minimum are attained at the extrema of the interval.
  - Case 2: At least one of the maximum or minimum is attained at an internal point.
- Case 1: Here $\max_{(a,b)} f(x) = \min_{(a,b)} f(x) = 0$, and this implies that $f(x)$ is constant & $f(x) = 0$ for all $x \in (a, b)$.

Proof of Rolle's Theorem

- Since $f(x)$ is continous, so by the Weirstraß Extreme Value Theorem the function admits a maximum and a minimum.
- There can be two cases:
  - Case 1: Both the maximum and minimum are attained at the extrema of the interval.
  - Case 2: At least one of the maximum or minimum is attained at an internal point.
- Case 1: Here $\max_{(a,b)} f(x) = \min_{(a,b)} f(x) = 0$, and this implies that $f(x)$ is constant & $f(x) = 0$ for all $x \in (a, b)$. So, $f'(x) = 0$ for all $x \in (a, b)$ and the theorem is verified.

Proof of Rolle's Theorem

- Since $f(x)$ is continous, so by the Weirstraß Extreme Value Theorem the function admits a maximum and a minimum.
- There can be two cases:
  - Case 1: Both the maximum and minimum are attained at the extrema of the interval.
  - Case 2: At least one of the maximum or minimum is attained at an internal point.
- Case 1: Here $\max_{(a,b)} f(x) = \min_{(a,b)} f(x) = 0$, and this implies that $f(x)$ is constant & $f(x) = 0$ for all $x \in (a, b)$. So, $f'(x) = 0$ for all $x \in (a, b)$ and the theorem is verified.
- Case 2: Let this point be c.

Proof of Rolle's Theorem

- Since $f(x)$ is continous, so by the Weirstraß Extreme Value Theorem the function admits a maximum and a minimum.
- There can be two cases:
  - Case 1: Both the maximum and minimum are attained at the extrema of the interval.
  - Case 2: At least one of the maximum or minimum is attained at an internal point.
- Case 1: Here $\max_{(a,b)} f(x) = \min_{(a,b)} f(x) = 0$, and this implies that $f(x)$ is constant & $f(x) = 0$ for all $x \in (a, b)$. So, $f'(x) = 0$ for all $x \in (a, b)$ and the theorem is verified.
- Case 2: Let this point be c. Since $f(x)$ is differentiable in (a, b), this point has to be a stationary point and hence $f'(c) = 0$.

Mean Value Theorem

Mean Value Theorem

Theorem (Mean Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continous function on a closed interval and f is differentiable at any point in the open interval (a, b). Then there exists at least one point $c \in (a, b)$ such that

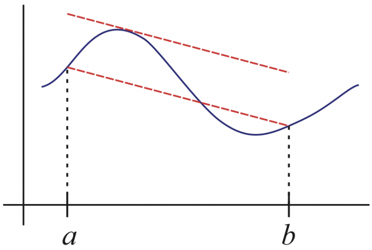
$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Mean Value Theorem

Theorem (Mean Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continous function on a closed interval and f is differentiable at any point in the open interval (a, b). Then there exists at least one point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



Proof of Mean Value Theorem

Proof of Mean Value Theorem

- We apply Rolle's Theorem to the following function

Proof of Mean Value Theorem

- We apply Rolle's Theorem to the following function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

Proof of Mean Value Theorem

- We apply Rolle's Theorem to the following function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

- h is continous and differentiable at all points where f is.

Proof of Mean Value Theorem

- We apply Rolle's Theorem to the following function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

- h is continous and differentiable at all points where f is.
- $h(a) = 0$ and $h(b) = 0$.

Proof of Mean Value Theorem

- We apply Rolle's Theorem to the following function
$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$
- h is continous and differentiable at all points where f is.
- $h(a) = 0$ and $h(b) = 0$.
- So, by Rolle's Theorem there exists at least one point $c \in (a, b)$ such that $h'(c) = 0$.

Proof of Mean Value Theorem

- We apply Rolle's Theorem to the following function

$$h(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

- h is continous and differentiable at all points where f is.
- $h(a) = 0$ and $h(b) = 0$.
- So, by Rolle's Theorem there exists at least one point $c \in (a, b)$ such that $h'(c) = 0$.
- This means $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. And the theorem is proved.

Generalized Rolle's Theorem

Generalized Rolle's Theorem

Theorem (Generalized Rolle's Theorem)

Let $f : [a, b] \to \mathbb{R}$ *be a continous function on a closed interval and it is differentiable at any point in the open interval* (a, b)*. Moreover there exists at least two points* $c, d \in [a, b]$ *such that* $f(c) = f(d)$*. Then there exists at least one point* $\xi \in (a, b)$ *such that* $f'(\xi) = 0$*.*

Generalized Rolle's Theorem

Theorem (Generalized Rolle's Theorem)

Let $f : [a, b] \to \mathbb{R}$ *be a continous function on a closed interval and it is differentiable at any point in the open interval* (a, b)*. Moreover there exists at least two points* $c, d \in [a, b]$ *such that* $f(c) = f(d)$*. Then there exists at least one point* $\xi \in (a, b)$ *such that* $f'(\xi) = 0$*.*

Geometrically, the picture is the same here!

Generalized Rolle's Theorem

Theorem (Generalized Rolle's Theorem)

Let $f : [a, b] \to \mathbb{R}$ be a continous function on a closed interval and it is differentiable at any point in the open interval (a, b). Moreover there exists at least two points $c, d \in [a, b]$ such that $f(c) = f(d)$. Then there exists at least one point $\xi \in (a, b)$ such that $f'(\xi) = 0$.

Geometrically, the picture is the same here!

We use this to prove Cauchy Mean Value Theorem.

Cauchy Mean Value Theorem

Cauchy Mean Value Theorem

Theorem (Cauchy Mean Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *be continous functions, defined on the closed interval* $[a, b]$ *and differentiable on the open interval* (a, b)*. Let* $g'(x) \neq 0$ *for all* $x \in (a, b)$*. Then there exists at least one point* $c \in (a, b)$ *such that*

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Cauchy Mean Value Theorem

Theorem (Cauchy Mean Value Theorem)

Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continous functions, defined on the closed interval $[a, b]$ and differentiable on the open interval (a, b). Let $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists at least one point $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

This gives us the Mean Value Theorem when $g(x) \equiv x$.

Proof of Cauchy Mean Value Theorem

Proof of Cauchy Mean Value Theorem

- We apply Rolle's theorem to the following function

$$h(x) = \Big(g(b) - g(a)\Big)f(x) - \Big(f(b) - f(a)\Big)g(x).$$

Proof of Cauchy Mean Value Theorem

- We apply Rolle's theorem to the following function

$$h(x) = \Big(g(b) - g(a)\Big)f(x) - \Big(f(b) - f(a)\Big)g(x).$$

- Notice $h(a) = g(b)f(a) - f(b)g(a) = h(b)$.

Proof of Cauchy Mean Value Theorem

- We apply Rolle's theorem to the following function

$$h(x) = \Big(g(b) - g(a)\Big) f(x) - \Big(f(b) - f(a)\Big) g(x).$$

- Notice $h(a) = g(b)f(a) - f(b)g(a) = h(b)$.
- By the Generalized Rolle's Theorem there exists at least one point $c \in (a, b)$ such that $h'(c) = 0$.

Proof of Cauchy Mean Value Theorem

- We apply Rolle's theorem to the following function

$$h(x) = \Big(g(b) - g(a)\Big) f(x) - \Big(f(b) - f(a)\Big) g(x).$$

- Notice $h(a) = g(b)f(a) - f(b)g(a) = h(b)$.
- By the Generalized Rolle's Theorem there exists at least one point $c \in (a, b)$ such that $h'(c) = 0$.
- This gives us $(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)$.

Proof of Cauchy Mean Value Theorem

- We apply Rolle's theorem to the following function

$$h(x) = \Big(g(b) - g(a)\Big) f(x) - \Big(f(b) - f(a)\Big) g(x).$$

- Notice $h(a) = g(b)f(a) - f(b)g(a) = h(b)$.
- By the Generalized Rolle's Theorem there exists at least one point $c \in (a, b)$ such that $h'(c) = 0$.
- This gives us $(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)$.
- Since $g'(c) \neq 0$, this implies $g(b) - g(a) \neq 0$, else if $g(a) = g(b)$ then by Rolle's theorem there exists at least one c such that $g'(c) = 0$.

Proof of Cauchy Mean Value Theorem

- We apply Rolle's theorem to the following function

$$h(x) = \Big(g(b) - g(a)\Big) f(x) - \Big(f(b) - f(a)\Big) g(x).$$

- Notice $h(a) = g(b)f(a) - f(b)g(a) = h(b)$.
- By the Generalized Rolle's Theorem there exists at least one point $c \in (a, b)$ such that $h'(c) = 0$.
- This gives us $(g(b) - g(a))f'(c) = (f(b) - f(a))g'(c)$.
- Since $g'(c) \neq 0$, this implies $g(b) - g(a) \neq 0$, else if $g(a) = g(b)$ then by Rolle's theorem there exists at least one c such that $g'(c) = 0$.
- Now dividing by $g'(c)$ and $g(b) - g(a)$ gives us the result.

L' Hôpital's Rule

L' Hôpital's Rule

Corollary (L' Hôpital's Rule)

Let $f : [a, b] \to \mathbb{R}$ *and* $g : [a, b] \to \mathbb{R}$ *be continous functions which are differentiable on the open interval* (a, b). *Let* $x_0 \in (a, b)$ *and suppose that* $g'(x) \neq 0$ *for all* $x \in (a, b) \setminus \{x_0\}$ *and*

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0.$$

If the limit of $\dfrac{f'}{g'}$ *exists at* x_0, *then*

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

L' Hôpital's Rule

Corollary (L' Hôpital's Rule)

Let $f : [a, b] \to \mathbb{R}$ and $g : [a, b] \to \mathbb{R}$ be continous functions which are differentiable on the open interval (a, b). Let $x_0 \in (a, b)$ and suppose that $g'(x) \neq 0$ for all $x \in (a, b) \setminus \{x_0\}$ and

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0.$$

If the limit of $\dfrac{f'}{g'}$ exists at x_0, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}.$$

This is a technique to evaluate limits of indeterminate forms.

Examples of L' Hôpital's Rule

Examples of L' Hôpital's Rule

- Let $f, g \to \mathbb{R}$ with $f(x) = \sin(x)$ and $g(x) = x$.

Examples of L' Hôpital's Rule

- Let $f, g \to \mathbb{R}$ with $f(x) = \sin(x)$ and $g(x) = x$.

$$\lim_{x \to 0} \frac{\sin x}{x}$$

Examples of L' Hôpital's Rule

- Let $f, g \to \mathbb{R}$ with $f(x) = \sin(x)$ and $g(x) = x$.

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

Examples of L' Hôpital's Rule

- Let $f, g \to \mathbb{R}$ with $f(x) = \sin(x)$ and $g(x) = x$.

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

- The result is false if $\lim_{x \to x_0} f(x) \neq 0$ or $\lim_{x \to x_0} g(x) \neq 0$.

Examples of L' Hôpital's Rule

- Let $f, g \to \mathbb{R}$ with $f(x) = \sin(x)$ and $g(x) = x$.

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

- The result is false if $\lim_{x \to x_0} f(x) \neq 0$ or $\lim_{x \to x_0} g(x) \neq 0$.
  - For instance, $f, g : [a, b] \to \mathbb{R}$, $f(x) = x^2 + 1$ and $g(x) = x + 2$.

Examples of L' Hôpital's Rule

- Let $f, g \to \mathbb{R}$ with $f(x) = \sin(x)$ and $g(x) = x$.

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\cos x}{1} = 1.$$

- The result is false if $\lim_{x \to x_0} f(x) \neq 0$ or $\lim_{x \to x_0} g(x) \neq 0$.
  - For instance, $f, g : [a, b] \to \mathbb{R}$, $f(x) = x^2 + 1$ and $g(x) = x + 2$. Applying the rule gives us

$$\lim_{x \to 0} \frac{x^2 + 1}{x + 2} = \frac{1}{2} \neq \lim_{x \to 0} \frac{2x}{1} = 2.$$

Remarks

Remarks

- Both the Rolle's Theorem and the Mean Value Theorem are existence results.

Remarks

- Both the Rolle's Theorem and the Mean Value Theorem are existence results.
- There is a more general Cauchy Mean Value Theorem for higher order derivatives, which again gives a higher order L' Hôpital's rule.

Thank you!