Differentiability:

"Given a curve $y=f(x)$ how can we compute the tangent line at a point $(x_0, f(x_0))$ of the curve?"

- This is difficult to do in general.
- The secant line bet^n two points $P_0 := (x_0, f(x_0))$ and $P_1 := (x_1, f(x_1))$ is always easy to compute. It is just the unique st. line that passes through the pts, given by

$$y = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0), \quad y_0 = f(x_0), \, y_1 = f(x_1).$$

- As P_0 and P_1 gets closer and closer, this secant line becomes closer and closer to the tangent. At the limit $P_1 \to P_0$, these two coincide.

<u>Def^n:</u> Given $f : (a,b) \to \mathbb{R}$ with $(a,b) \neq \phi$, and a point $x_0 \in I$, we say that f is <u>differentiable at x_0</u> if the following limit exists and is finite: $\lim\limits_{x \to x_0} \dfrac{f(x) - f(x_0)}{x - x_0}$.

This limit is called the <u>derivative of f at x_0</u>, denoted by $f'(x_0)$.

- If this limit doesn't exist or is not finite, then we say that f is <u>not differentiable at the point x_0</u>.

- $x_0 \in (a,b)$ is always a cluster point for (a,b) so the limit is well-defined.

- Usually, we don't define the derivative of a fn: $[a,b] \to \mathbb{R}$ at the end points a, b since there the idea of a tangent line doesn't make sense.

eg: • $f(x) = x$ is always differentiable on $\mathbb{R}$.

$$\lim_{x \to x_0} \frac{x - x_0}{x - x_0} = 1 = f'(x) \quad \forall x \in \mathbb{R}.$$

• $f(x) = |x|$ is not differentiable at $x_0 = 0$.

$$\lim_{x \to 0^+} \frac{|x| - 0}{x - 0} = 1, \quad \lim_{x \to 0^-} \frac{|x| - 0}{x - 0} = -1.$$

- The quotient $\frac{f(x) - f(x_0)}{x - x_0}$ is called the Newton-Quotient.

Geometrically, it is the slope of the secant line through $(x_0, f(x_0))$ and $(x, f(x))$. So, geometrically the derivative is the <u>slope of the tangent line</u> to the graph of f at the point $(x_0, f(x_0))$.

eg: $f'(x) = 2x$ for $f(x) = x^2$.

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x \to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x \to x_0} (x + x_0) = 2x_0.$$

• Equivalent defn of derivative is $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$.

eg: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h}$$

$$= \lim_{h \to 0} \left[\frac{\sin h/2}{h/2} \cos\left(x + h/2\right) \right]$$

$$= \lim_{h \to 0} \frac{\sin h/2}{h/2} \lim_{h \to 0} \cos\left(x + h/2\right)$$

$$= 1 \cdot \cos x = \cos x.$$

$$\left[\begin{array}{l} \sin(x-h) - \sin x = \\ \quad 2 \sin \frac{x+h-x}{2} \cos \frac{x+h+x}{2} \\ \lim_{x \to 0} \frac{\sin x}{x} = 1. \end{array} \right]$$

<u>Theorem:</u> If $f: (a,b) \to \mathbb{R}$ is differentiable at $x_0 \in (a,b)$, then f is cont. at x_0.

<u>Proof:</u> f is cont. at x_0 iff $\lim_{x\to x_0} f(x) = f(x_0)$

iff $\lim_{x\to x_0} (f(x) - f(x_0)) = 0$.

Now, $\lim_{x\to x_0}(f(x)-f(x_0)) = \lim_{x\to x_0}\left[\frac{f(x)-f(x_0)}{(x-x_0)}(x-x_0)\right]$

$= \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} \lim_{x\to x_0}(x-x_0)$

$= f'(x_0) \cdot 0 = 0$. //.

- The reverse is NOT true. eg $f(x) = |x|$ is cont. on $\mathbb{R}$ but not differentiable at 0.
- The theorem implies, if a fn is discontinuous at x_0, then it is NOT differentiable at x_0.

  eg: $\frac{1}{x}, \frac{1}{x^2}$ are not differentiable at 0.
- Consider $f: \mathbb{R} \to \mathbb{R}$, $f(x) = \begin{cases} x^3 & , x \leq 1 \\ 2-x^2 & , x > 1 \end{cases}$

  $f(x)$ is cont. It is obvious for $(-\infty, 1)$ and $(1, +\infty)$.

Also, $\lim_{x\to 1+} f(x) = \lim_{x\to 1+}(2-x^2) = 1$, $\lim_{x\to 1-} f(x) = 1$.

And $\lim_{x\to 1} f(x) = 1 = f(1)$.

But, the function is NOT differentiable at 1.

$$\lim_{x\to 1+}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1+}\frac{2-x^2-1}{x-1} = -2.$$

$$\lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = 3. \quad //.$$

<u>Rules of differentiation:</u> Let $f: (a,b) \to \mathbb{R}$ and $g: (a,b) \to \mathbb{R}$ be two fns differentiable at $x_0 \in (a,b)$ and $c \in \mathbb{R}$, then

- The fn $h(x) := c f(x)$ is diff. at x_0, $h'(x_0) = cf'(x_0)$.
- The fn $h(x) := f(x) + g(x)$ - .. at x_0, $h'(x) = f'(x_0) + g'(x_0)$.

- The fn $h(x) := f(x)g(x)$ is diff. at x_0 and
$$(fg)'(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0).$$
- If $g(x_0) \neq 0$, then the fn $h(x) := \frac{f(x)}{g(x)}$ is diff at x_0 and
$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{(g(x_0))^2}.$$

Proofs: observe:
- $$\frac{f(x)+g(x)-f(x_0)-g(x_0)}{x-x_0} = \frac{f(x)-f(x_0)}{x-x_0} + \frac{g(x)-g(x_0)}{x-x_0}.$$
- $$\frac{f(x)g(x)-f(x_0)g(x_0)}{x-x_0} = \frac{f(x)-f(x_0)}{x-x_0}\cancel{g(x)}\, g(x) + \frac{g(x)-g(x_0)}{x-x_0}\cdot f(x_0)$$
- $$\frac{\frac{1}{g(x)} - \frac{1}{g(x_0)}}{x-x_0} = \frac{1}{g(x)g(x_0)} \frac{g(x_0)-g(x)}{x-x_0} \Rightarrow h'(x) = \frac{1}{g'(x)} = \frac{g'(x)}{g(x)g(x_0)}$$

Combine this with previous result.

Chain Rule: Let $(a,b) \subseteq \mathbb{R}$, $(c,d) \subseteq \mathbb{R}$ and $f : (a,b) \to \mathbb{R}$ and $g : (c,d) \to \mathbb{R}$ be such that $g((c,d)) \subseteq (a,b)$ and $x_0 \in (c,d)$. If g is diff. at x_0 and f is diff at $y_0 = g(x_0) \in (a,b)$, then $h(x) := f(g(x))$ is diff at x_0 and $h'(x_0) = f'(g(x_0)) \cdot g'(x_0)$.

Proof: Note whenever $g(x) \neq g(x_0)$, we have
$$\frac{f(g(x)) - f(g(x_0))}{x-x_0} = \frac{f(g(x)) - f(g(x_0))}{g(x)-g(x_0)} \cdot \frac{g(x)-g(x_0)}{x-x_0}$$

We assume that this is true at least near x_0. (We prove this milder version only.) Define $y := g(x)$ and $y_0 = g(x_0)$).

By cont of $g(x)$ at x_0 we have as $x \to x_0$, $y \to y_0$.

Thus, $$\lim_{x\to x_0} \frac{f(g(x)) - f(g(x_0))}{g(x)-g(x_0)} = \lim_{y \to y_0} \frac{f(y)-f(y_0)}{y-y_0} = f'(y_0) = f'(g(x_0)).$$

//

eg: $h: \mathbb{R} \to \mathbb{R}$, $h(x) = e^{f(x)} \Rightarrow h'(x) = f'(x)\, e^{f(x)}$.

$$h(x) = e^{(f(x))^2} \Rightarrow h'(x) = (f(x)^2)'\, e^{(f(x))^2}$$

$$= f'(x)\, 2f(x)\, e^{(f(x))^2}.$$

Derivative of the Inverse Function: Let (a,b) & $(c,d) \subseteq \mathbb{R}$, $f:(a,b) \to (c,d)$ be cont. and invertible. We consider $f^{-1}:(c,d) \to (a,b)$.

If f is diff. at $x_0 \in (a,b)$ and $f'(x_0) \neq 0$, then f^{-1} is diff at $y_0 := f(x_0)$ and $(f^{-1})'(y_0) = \dfrac{1}{f'(x_0)} = \dfrac{1}{f'(f^{-1}(y_0))}$.

Proof: Write $x = f^{-1}(y)$, $x_0 = f^{-1}(y_0)$. Now $y \to y_0 \Rightarrow x \to x_0$.

$$\lim_{y \to y_0} \frac{f^{-1}(y) - f^{-1}(y_0)}{y - y_0} = \lim_{x \to x_0} \frac{x - x_0}{f(x) - f(x_0)} = \frac{1}{f'(x_0)} = \frac{1}{f'(f^{-1}(y_0))}. \ //.$$

eg: $g:(0,+\infty) \to \mathbb{R}$, $g(x) = \ln x$.

Let $g = f^{-1}$, $f: \mathbb{R} \to (0,+\infty)$, $f(x) = e^x$.

Now, $g'(x) = (f^{-1}(x))' = \dfrac{1}{f'(f^{-1}(x))} = \dfrac{1}{e^{f^{-1}(x)}} = \dfrac{1}{e^{\ln x}} = \dfrac{1}{x}$. //.

eg: $f: \mathbb{R} \to (-\pi/2, \pi/2)$, $f(x) = \arctan(x/a)$, $a > 0$.

$g: (-\pi/2, \pi/2) \to \mathbb{R}$, $g(x) = a \tan x$, $g^{-1} = f$.

$(g^{-1})'(x) = \dfrac{1}{g'(g^{-1}(x))}$, $g'(x) = a \sec^2 x = a(\tan^2 x + 1)$.

So, $g'(g^{-1}(x)) = a(\tan^2((g'(x))) + 1) = a(\tan^2(\arctan(x/a)) + 1)$

$$= a((x/a)^2 + 1).$$

$$= \frac{x^2 + a^2}{a}.$$

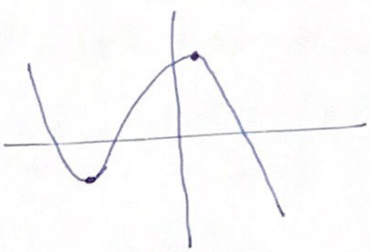
Thus, $f'(x) = (g^{-1})'(x) = \dfrac{1}{\frac{x^2+a^2}{a}} = \dfrac{a}{x^2+a^2}$. //.

Def^n: Given a fn $f:(a,b)\to\mathbb{R}$ diff at $x_0\in(a,b)$. We say that x_0 is a <u>stationary point</u> if $f'(x_0)=0$.

eg: 0 is the unique stationary pt. of $f(x)=x^3$.

Def^n: Given $f:(a,b)\to\mathbb{R}$, we say that $x_0\in(a,b)$ is a <u>local max^n pt.</u> (resp. a <u>local min^m pt.</u>) if there exists $\delta>0$ s.t. x_0 is a max^m pt. (resp. a min^m pt) for $f(x)$ in $(x_0-\delta, x_0+\delta)\subset(a,b)$.

eg: $f(x)=-x^3-2x^2+x+2$, the pt. $\frac{2+\sqrt{7}}{3}$ is a local max^m and $\frac{2-\sqrt{7}}{3}$ is a local min^m.



Thm: Given a fn: $(a,b)\to\mathbb{R}$, diff at $x_0\in(a,b)$. If x_0 is a local max^m pt. or a local min^m pt., then $f'(x_0)=0$.

Proof: (For local max^m pt.) Note, since $f(x)$ is diff. at x_0,

$$\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}=\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}.$$

Since, x_0 is a local max^m pt, there exists $\delta>0$ s.t. $f(x)\le f(x_0)$ for all $x\in(x_0-\delta, x_0+\delta)$.

Then, $\frac{f(x)-f(x_0)}{x-x_0}\le 0$, $x\to x_0^+$ and

$\frac{f(x)-f(x_0)}{x-x_0}\geq 0$ as $x\to x_0^-$.

This implies, $\lim_{x\to x_0^+}\frac{f(x)-f(x_0)}{x-x_0}\le 0$ & $\lim_{x\to x_0^-}\frac{f(x)-f(x_0)}{x-x_0}\geq 0$.

Thus, $f'(x_0)=0$. //

- The converse is NOT true. eg. $f(x) = x^3$, 0 is a stationary point, but 0 is neither a local max$^{\underline{m}}$ nor a local min$^{\underline{m}}$.

In fact, for any $\delta > 0$, take $x_1 = -\delta/2$, $x_2 = \delta/2$ so that, $x_1, x_2 \in (0-\delta, 0+\delta) = (\delta, \delta)$ but $f(x_1) < f(0) < f(x_2)$.

- The theorem doesn't say that every local max$^{\underline{m}}$ or min$^{\underline{n}}$ is a stationary point. This is only true when the fn is diff at that pt. eg: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = |x|$ has max$^{\underline{m}}$ at origin, but not diff. there.