Eigenvalues and Eigenvectors:

- We are now looking for sol^n of $Ax = \lambda x$. ~ we will again use determinants.
- This is what is called an <u>eigenvalue</u> problem. (google...)

$$Ax = \lambda x \text{ is equivalent to } (A - \lambda I)x = 0.$$

- The vector x is in the nullspace of $A - \lambda I$.
- The number λ is choosen so that $A - \lambda I$ has a nullspace. ~ non-zero x.
- In short, $A - \lambda I$ must be singular.

<u>Theorem:</u> The no. λ is an <u>eigenvalue</u> of A iff $A - \lambda I$ is singular.

<u>i.e.</u> $\det(A - \lambda I) = 0$. This is called the <u>characteristic eq^n</u>.

Each λ is associated with <u>eigenvectors</u> x s.t. $(A - \lambda I)x = 0$.

$\det(A - \lambda I)$ is called the <u>characteristic polynomial</u>.

e.g. Let $A = \begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$, then $A - \lambda I = \begin{pmatrix} 4-\lambda & -5 \\ 2 & -3-\lambda \end{pmatrix}$.

$\det(A - \lambda I) = \lambda^2 - \lambda - 2$. So, eigenvalues are $\lambda = -1, 2$.

For $\lambda = -1$, we get, $(A - \lambda_1 I)x = 0 \Rightarrow \begin{pmatrix} 5 & -5 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\Rightarrow \begin{pmatrix} y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} := x_1.$$

Similarly, $x_2 = \begin{pmatrix} 5 \\ 2 \end{pmatrix}$.

<u>Some good matrices w.r.t. eigenvalues:</u>

1. <u>Diagonal matrices:</u> Eigenvectors of 2×2 ~~giv~~ are $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$.
   eg: $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = A$.
2. <u>Projection matrices:</u> The eigenvalues are 1 or 0.
   When it is 1, it projects onto ~~int~~ itself.
   When it is 0, it projects to the zero vector.
3. <u>Triangular matrices:</u> Eigenvalues are on the main diagonal.

* So we might want to change a matrix into a diagonal / triangular matrix without changing its eigenvalues.

Ex: Determine if $v = \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix}$, $u = \begin{pmatrix} -1 \\ 2 \\ 1 \end{pmatrix}$ are eigenvectors of

$$A = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix}.$$

Sol^n: Let us compute $Av = \begin{pmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{pmatrix} \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -6 \\ 0 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} -3 \\ 0 \\ 1 \end{pmatrix} = 2v.$

So, v is an eigenvector corresponding to eigenvalue $\lambda = 2$.

IIIly, find Au and see if we get some scalar λ s.t. $Au = \lambda u$. //.

Def^n: The subspace $N(A - \lambda I)$ is called the <u>eigenspace</u> of A corresponding to λ.

Ex: It is known that $\lambda = 3$ is an eigenvalue of $A = \begin{pmatrix} 11 & -4 & -8 \\ 4 & 1 & -4 \\ 8 & -4 & -5 \end{pmatrix}$.

Find the eigenspace of A corresponding to $\lambda = 3$.

Sol^n: Compute $A - \lambda I = \begin{pmatrix} 8 & -4 & -8 \\ 4 & -2 & -4 \\ 8 & -4 & -8 \end{pmatrix}$. Now we find $N(A - \lambda I)$.

The row reduced echelon form of A is obtained after doing the following transformation: $R_1 \updownarrow R_2$, $-2R_1 + R_2$, $-2R_1 + R_3$ and we obtain,

$$R = \begin{pmatrix} 4 & -2 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$\uparrow$ pivot col^m free col^{m}s.

So, $N(A - \lambda I) = t_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t_2 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}$.

$\hookrightarrow$ the system is $2t_1 - t_2 - 2t_3 = 0$

So, the eigenspace is, $N(A - \lambda I) = \text{span}\left\{ \underbrace{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}}_{\text{linearly independent, so this is a basis as well.}} \right\}$. //.

Theorem: Let $v_1, v_2, \ldots, v_k$ be eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then $\{v_1, \ldots, v_k\}$ is a linearly independent set.

Proof: Suppose $\{v_1, \ldots, v_k\}$ is linearly dependent and $\{\lambda_1, \ldots, \lambda_k\}$ are distinct. Then at least one eigenvector say v_{p+1} is a linear combination of $v_1, \ldots, v_p$ and $\{v_1, \ldots, v_p\}$ is linearly independent.

So, $v_{p+1} = c_1 v_1 + \cdots + c_p v_p$ for scalars c_i.

$\Rightarrow A v_{p+1} = c_1 A v_1 + \cdots + c_p A v_p$

$\Rightarrow \lambda_{p+1} v_{p+1} = c_1 \lambda_1 v_1 + \cdots + c_p \lambda_p v_p$. —— 1

Again, $\lambda_{p+1} v_{p+1} = c_1 \lambda_{p+1} v_1 + \cdots + c_p \lambda_{p+1} v_p$ —— 2

1 − 2 $\Rightarrow \quad 0 = c_1(\lambda_1 - \lambda_{p+1}) v_1 + \cdots + c_p(\lambda_p - \lambda_{p+1}) v_p$.

Since $\{v_1, \dots, v_p\}$ is linearly indep. so, $c_i(\lambda_i - \lambda_{p+1}) = 0$.

$\Rightarrow c_i = 0$ since $\lambda_i \neq \lambda_j$ for $i \neq j$.

$\Rightarrow v_{p+1} = 0$, which is a contradiction.
since eigenvectors are non-zero.
//.

Remark: The matrix A is invertible iff $\lambda = 0$ is not an eigenvalue of A.
(If $\lambda = 0$ is an eigenvalue then, there exist some x s.t. $Ax = 0.x = 0$, so x is in the nullspace of A.)

Characteristic Polynomial:

Theorem: The characteristic polynomial $p(\lambda) = \det(A - \lambda I)$ of an $n \times n$ matrix A is an nth degree polynomial.

Proof: Use induction on n and expand along the 1st row.//.

Theorem: Suppose A is an $n \times n$ matrix and has n distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Let v_i be an eigenvector of A corresponding to λ_i.
Then $\{v_1, v_2, \dots, v_n\}$ is a basis for $\mathbb{R}^n$.

Remark: If A has distinct eigen~~vecto~~values we always get a basis of $\mathbb{R}^n$ consisting of eigenvectors of A.

But not every matrix has a set of eigenvectors that forms a basis for $\mathbb{R}^n$.

– If A has some repeated eigenvalues, then only in some cases we get a basis for $\mathbb{R}^n$.

Ex: Find the eigenvalues of A and bais for each eigenspace, where $A = \begin{pmatrix} 2 & 0 & 0 \\ 4 & 2 & 2 \\ -2 & 0 & 1 \end{pmatrix}$. Does $\mathbb{R}^3$ have a bais of eigenvectors of A?

Solⁿ: $P(\lambda) = \det(A - \lambda I) = \lambda^3 - 5\lambda^2 + 8\lambda - 4 = (\lambda - 1)(\lambda - 2)^2$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$ where λ_2 is repeated.

For $\lambda_1 = 1$, $(A - \lambda I)x = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 4 & 1 & 2 \\ -2 & 0 & 0 \end{pmatrix} x = 0 \Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} x = 0$

free colᵐⁿ.

we will get $N(A - \lambda_1 I) = \text{span}\left\{\begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}\right\}$.

$v_1 = \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$ is an eigenvector of A with eigenvalue $\lambda_1 = 1$.

For $\lambda_2 = 2$ we will get, $A - 2I \sim \begin{pmatrix} -2 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ so, $\text{rank}(A - 2I) = 1$

$\Rightarrow \text{nullity}(A - 2I) = 2$.

we will get, $N(A - \lambda_2 I) = \text{span}\{v_2, v_3\}$ where, $v_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$, $v_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$.

The eigenspace is $\{v_1, v_2, v_3\}$ which forms a bais for $\mathbb{R}^3$. //.

Defⁿ: Let $p(\lambda)$ be the characteristic polynomial of $A \in \mathbb{R}^{n \times n}$ and,

$P(\lambda) = (\lambda - \lambda_1)^{k_1}(\lambda - \lambda_2)^{k_2} \cdots (\lambda - \lambda_p)^{k_p}$. The exponents k_i's are called the algebraic multiplicity of the eigenvalue λ_i.

The $\dim N(A - \lambda_i I)$ is called the geometric multiplicity of λ_i, denoted by g_i.

eg. In the previous ex, we had $g_1 = 1$ and $g_2 = 2$; $k_1 = 1$ and $k_2 = 2$.

Remark: In general, $g_i \le k_i$.