Indian Institute of Information Technology (IIIT) Manipur

End Semester Examination, February 2023

Course Title: **Mathematics I** | Course Code: **MA1011/MA101**

Semester: I | Maximum Marks: 100

Date of Examination: 24 February 2023 | Time: 3 hours

Total Number of Pages: 3

Part A (10×2 marks $= 20$ marks)

1. Given that $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix} A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, find A^{-1}. (**2 marks**)

2. Given that $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix} A = \begin{pmatrix} 23 & 0 & 0 \\ 0 & 0 & 23 \\ 0 & 23 & 0 \end{pmatrix}$, find the value of $\det(A)$. (**2 marks**)

3. Is there a matrix A for which $(1 \quad 2 \quad 1)^T$ is a basis for the column space and $(1 \quad 1 \quad 1)^T$ is a basis for the nullspace? If yes, what is A? if no, why does no such A exist? (**1+1 marks**)

4. Find the supremum and infimum of the following set (**1+1 marks**)

$$\left\{ 2(-1)^{n+1} + (-1)^{\frac{n(n+1)}{2}} \left(2 + \frac{3}{n}\right) : n \in \mathbb{N} \right\}.$$

5. Is the series $\sum_{i=1}^{\infty} \sin i$ convergent? Justify your answer. (**1+1 marks**)

6. Write the statement of Taylor's Theorem. (**2 marks**)

7. If a function f has the property that for all real numbers x, we have $3 - |x| \leq f(x) \leq 3 + |x|$, then from this conclude $f(x) \to$ ______ as $x \to$ ______ . (Fill in both the blanks for full credit.) (**2 marks**)

8. Evaluate $\int_0^3 f(x)\mathrm{d}x$, where $f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2. \end{cases}$ (**2 marks**)

9. Find an interval $[a, b]$ on which $f(x) = x^4 + x^3 - x^2 + x - 2$ satisfies the hypothesis of the Rolle's theorem. (**2 marks**)

10. State the Cauchy convergence criterion and the monotone convergence theorem. (**1+1 marks**)

Part B (5×16 marks $= 80$ marks)

11. If $f(x) = x^{2/3}$, $a = -1$ and $b = 8$,

    (I) Show that there is no point c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. (**8 marks**)

    (II) Explain why the result in part (I) does not contradict the Mean-Value Theorem. (**8 marks**)

12. (A) Given $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & -1 \end{pmatrix}$.

(I) Apply the Gram-Schmidt process to the columns of the matrix A in the order that they occur in the matrix. Use this to write $A = QU$, where Q is a matrix with orthonormal columns and R is an upper triangular matrix. **(4+4 marks)**

(II) Compute the matrix of the projection onto the column space of A. What is the distance of the point $(1, 1, 1, 0)$ to this column space? **(4+4 marks)**

OR

(B) Given $A = \begin{pmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{pmatrix}$.

(I) Find the eigenvalues of the matrix A. **(6 marks)**

(II) Diagonalize the matrix A. **(10 marks)**

13. (A) Given that $f : (a, b) \to \mathbb{R}$ is a differentiable function.

(I) Prove that $f(x)$ is increasing on (a, b) iff $f'(x) \geq 0$ for all $x \in (a, b)$. **(6 marks)**

(II) Prove that if $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is strictly increasing. **(6 marks)**

(III) Is the reverse implication of part (II) true? If yes, why? If no, why not? **(4 marks)**

OR

(B) (I) Show that the series $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$. **(8 marks)**

(II) Show that the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ is convergent for $-1 < x \leq 1$. **(8 marks)**

14. (A) (I) Given that f is continuous on $[a, b]$, $a > 0$ and differentiable on (a, b). Show that if

$$\frac{f(a)}{a} = \frac{f(b)}{b},$$

then there exists an $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$. **(6 marks)**

(II) Given that f is continuous on $[0, 2]$ and twice differentiable on $(0, 2)$. Show that if

$$f(0) = 0, \quad f(1) = 1 \quad \text{and} \quad f(2) = 2,$$

then there exists an $x_0 \in (0, 2)$ such that $f''(x_0) = 0$. **(6 marks)**

(III) Given that $f : \mathbb{R} \to \mathbb{R}$ is $n + 1$ times differentiable on $\mathbb{R}$. Prove that for every $x \in \mathbb{R}$ there is a $\theta \in (0, 1)$ such that

$$f(x) = f(0) + xf'(x) - \frac{x^2}{2} f''(x) + \cdots + (-1)^{n+1} \frac{x^n}{n!} f^{(n)}(x) + (-1)^{n+2} \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x),$$

and

$$f\left(\frac{x}{1+x}\right) = f(x) - \frac{x^2}{1+x}f'(x) + \cdots + (-1)^n \frac{x^{2n}}{(1+x)^n}\frac{f^{(n)}(x)}{n!}$$
$$+ (-1)^{n+1}\frac{x^{2n+2}}{(1+x)^{n+1}}\frac{f^{(n+1)}\left(\frac{x+\theta x^2}{1+x}\right)}{(n+1)!}, \quad x \neq -1.$$

(**2+2 marks**)

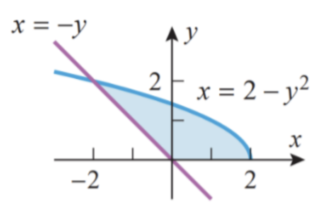
OR

(B) (I) Sketch the region whose area is represented by the following definite integrals, and evaluate the integral using an appropriate formula: (**5+5 marks**)

(a) $\int_{-1}^{2} (x+2)\mathrm{d}x$,

(b) $\int_{0}^{1} \sqrt{1-x^2}\mathrm{d}x$.

(II) Find the area of the shaded region in the figure given below. (**6 marks**)



15. (A) (I) Find an antiderivative of $f : \mathbb{R} \to \mathbb{R}$ defined by $f(\theta) = \sin^4\theta\cos^2\theta$. (**5 marks**)

(II) Evaluate the integral $\int_{0}^{\pi/2} \frac{1+\cos\theta}{3-\cos\theta}\mathrm{d}\theta$. (**5 marks**)

(III) Find the values of $\alpha \in \mathbb{R}$ for which the integral $\int_{1}^{\infty} x^{\alpha}\mathrm{d}x$ converges. (**6 marks**)

OR

(B) (I) Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis. (**8 marks**)

(II) Find the exact arc length of the curve $y = 3x^{3/2} - 1$ from $x = 0$ to $x = 1$. (**8 marks**)