The Fundamental Theorem of Calculus:

Def^n: Let $f:[a,b]\to\mathbb{R}$ be a real-valued fn. The fn $F:[a,b]\to\mathbb{R}$ is an antiderivative of f on $[a,b]$ if F is cont. on $[a,b]$ and diff. on (a,b) s.t. for all $x\in(a,b)$, we have $\boxed{F'(x)=f(x)}$.

eg: $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^x$ then $F(x)=e^x$.

$f:\mathbb{R}\to\mathbb{R}$, $f(x)=\cos x$, then $F(x)=\sin x$

$f:\mathbb{R}\to\mathbb{R}$, $f(x)=x$, then $F(x)=\frac{x^2}{2}$.

$f:\mathbb{R}\to\mathbb{R}$, $f(x)=x^n$, then $F(x)=\dfrac{x^{n+1}}{n+1}$.

- Antiderivatives are not unique!!

Thm: If f is cont. on $[a,b]$ and $f'(x)=0$ $\forall x\in(a,b)$, then $f(x)=c$ for some constant c, $\forall x\in[a,b]$.

Proof: By the MVT, we have, $f(x)-f(a)=f'(c)\cdot(x-a)$ where $x\in(a,b]$ and we apply MVT on $[a,x]$, where $a<c<x$.

Let $c:=f(a)$, $f'(c)=0 \Rightarrow f(x)-f(a)=f(x)-c=0$

$\Rightarrow f(x)=c \;\; \forall x\in(a,b]$. //

Corollary: Let F, G be any two antiderivatives of a fn $f:[a,b]\to\mathbb{R}$, then F and G differ by a constant.

Proof: Consider, $H(x)=F(x)-G(x)$ and then apply the previous result. //

eg: Consider $f:\mathbb{R}\to\mathbb{R}$, $f(x)=|x| = \begin{cases} x, & \text{if } x\geq 0 \\ -x, & \text{if } x<0 \end{cases}$

$$F(x)=\begin{cases} \frac{x^2}{2}+c_1, & \text{if } x>0 \\ -\frac{x^2}{2}+c_2, & \text{if } x<0 \end{cases}, \quad c_1, c_2\in\mathbb{R}.$$

For F to be differentiable at 0 we need it to be continuous. $\lim_{x\to 0^-} F(x) = \lim_{x\to 0^-} -\frac{x^2}{2} = 0$, $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+}\left(\frac{x^2}{2}+C_1\right) = C_1$.

So we want $C_1 = 0$ as well.

Thus, $F(x) = \begin{cases} x^2/2 & \text{if } x \geq 0 \\ -x^2/2 & \text{if } x < 0 \end{cases}$ is differentiable on $\mathbb{R}$.

So, even if a fn is not diff. at a point, an antiderivative can be defined sometimes.

- Antiderivatives do not always exist. However, any cont. fn does have an antiderivative (proof later).

eg: $h : \mathbb{R} \to \mathbb{R}$, $h(x) = \begin{cases} -1, & \text{if } x < 0 \\ 1, & \text{if } x \geq 0. \end{cases}$

Any antiderivative of h, say H looks like,

$$H(x) = \begin{cases} -x + C_1, & \text{if } x < 0 \\ x + C_2, & \text{if } x > 0 \\ k, & \text{if } x = 0. \end{cases}$$

WLOG, assume $C_2 = 0$. H must be cont. on $\mathbb{R}$ for it to be diff.
So, at $x = 0$ we need, $C_1 = 0$ and $k = 0$. Thus,

$$H(x) = \begin{cases} -x, & \text{if } x < 0 \\ x, & \text{if } x \geq 0 \end{cases} = |x|.$$

So, H is just the absolute value fn (upto a constant) which we know is not diff. at $x = 0$. Thus, h has no antiderivative on $\mathbb{R}$.

__Def^n:__ Let $f : [a,b] \to \mathbb{R}$ and let F be an antiderivative of f on $[a,b]$. The __indefinite integral of f on $[a,b]$__, denoted by $\int f(x)\,dx$, is defined by $\int f(x)\,dx = F(x) + C$, $C \in \mathbb{R}$.

<u>Theorem</u>: Let $f, g : I \to \mathbb{R}$ be a real-valued fn on an interval $I \subset \mathbb{R}$, and let $\alpha \in \mathbb{R}$. If $F, G : I \to \mathbb{R}$ are antiderivatives for f, g resp, then

(a) $H : I \to \mathbb{R}$, defined by $H(x) = \alpha F(x)$ is an antiderivative for the fn $h : I \to \mathbb{R}$ defined by $h(x) = \alpha f(x)$.

(b) $H : I \to \mathbb{R}$, defined by $H(x) = F(x) + G(x)$ is an antiderivative for the fn $h : I \to \mathbb{R}$ defined by $h(x) = f(x) + g(x)$.

<u>Definite Integrals</u>:

<u>Defn</u>: A <u>partition</u> of an interval $[a,b]$ is a finite collection of points in $[a,b]$, one of which is a and one of which is b.

- Let $a = x_0 < x_1 < \cdots < x_n = b$, then $P \doteq (x_0, x_1, \dots, x_n)$ is a partition, each interval $[x_{i-1}, x_i]$ is called a <u>subinterval</u> of the partition.

<u>Defn</u>: Let $f : [a,b] \to \mathbb{R}$ be a bdd. fn and let $P = (x_0, \dots, x_n)$ be a partition of $[a,b]$. The <u>upper sum</u> $U_{f,P}$ and the <u>lower sum</u> $L_{f,P}$ of f w.r.t. P are defined as,

$$U_{f,P} := \sum_{i=1}^{n} (x_i - x_{i-1}) M_i \quad , \text{ where } M_i := \sup_{x \in [x_{i-1}, x_i]} f(x)$$

$$\text{and } L_{f,P} := \sum_{i=1}^{n} (x_i - x_{i-1}) m_i , \quad \text{where } m_i := \inf_{x \in [x_{i-1}, x_i]} f(x),$$

for $i = 1, 2, \dots, n$.

- Here $(x_i - x_{i-1})$ is the <u>width</u> of the i^{th} <u>sub-interval</u> $[x_{i-1}, x_i]$. So, $(x_i - x_{i-1}) M_i$ and $(x_i - x_{i-1}) m_i$ are the areas of rectangles of heights M_i and m_i resp.

• f needs to be bdd, otherwise we will have either $M_i = +\infty$ or $m_i = -\infty$ for some i.

<u>Defn</u>: Let $f : [a,b] \to \mathbb{R}$ be a bdd fn. If $\sup_P L_{f,P} = \inf_P U_{f,P}$, we say that f is <u>integrable</u> over $[a,b]$ and we call this number the <u>integral</u>, denoted by $\int_a^b f\, dx$.

• Geometrically, the integral represents the (signed) area under the curve of f betn a and b.

<u>Defn</u>: Let f be integrable on $[a,b]$. We define $\int_a^b f(x)\, dx := \int_a^b -f(x)\, dx$.

<u>eg</u>: Let $f(x) = c \in \mathbb{R}$, a constant. For any subinterval of P of $[a,b]$ we have $m_i = M_i = c$.

Thus, $U_{f,P} = \sum_{i=1}^{n} (x_i - x_{i-1}) M_i = \sum_{i=1}^{n} (x_i - x_{i-1})\, c$

$$= c(x_1 - x_0 + x_2 - x_1 + \cdots + x_n - x_{n-1})$$

$$= c(x_n - x_0) = c(b-a).$$

Illy, $L_{f,P} = c(b-a)$. So, $\int_a^b c\, dx = c(b-a)$. //.

• Not all bdd fns are integrable.

<u>eg</u>: Let $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 1, & \text{if } x \in \mathbb{Q}^c. \end{cases}$

For any sub-interval, f takes both the values 0 and 1.

So, $\inf_{x \in [x_{i-1}, x_i]} f(x) = 0$, $\sup_{x \in [x_{i-1}, x_i]} f(x) = 1$, $\forall\, 1 \le i \le n$.

Thus, $L_{f,P} = 0$ and $U_{f,P} = 1$. So, f is not integrable. //.

Fundamental Theorem of Calculus: Suppose $f : [a,b] \to \mathbb{R}$ is integable and has an antiderivative F. Then, we have

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof: Let P be a partition of $[a,b]$. By the MVT, $\forall\, i = 1, \dots n$ $\exists\, t_i \in (x_{i-1}, x_i)$ s.t.

$$\begin{aligned} F(x_i) - F(x_{i-1}) &= F'(t_i)(x_i - x_{i-1}) \\ &= f(t_i)(x_i - x_{i-1}). \end{aligned}$$

We have, ~~Thus~~,

$$m_i(x_i - x_{i-1}) \le f(t_i)(x_i - x_{i-1}) \le M_i(x_i - x_{i-1})$$

$$\Rightarrow m_i(x_i - x_{i-1}) \le F(x_i) - F(x_{i-1}) \le M_i(x_i - x_{i-1}).$$

$$\Rightarrow \sum_{i=1}^{n} m_i(x_i - x_{i-1}) \le \sum_{i=1}^{n}\left(F(x_i) - F(x_{i-1})\right) \le \sum_{i=1}^{n} M_i(x_i - x_{i-1})$$

$$\Rightarrow L_{f,P} \le F(b) - F(a) \le U_{f,P}. \quad \text{——} \ (1)$$

So, $F(b) - F(a)$ is an upper bound for the lower sums, but since the $\sup_P L_{f,P}$ is the least upper bound, we have

$$\sup_P L_{f,P} \le F(b) - F(a).$$

By a similar logic we have, $F(b) - F(a) \le \inf_P U_{f,P}$.

Since f is integable we have $\sup_P L_{f,P} = \inf_P U_{f,P}$ and hence the result follows. //.

eg: $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \sin x$.

Since f is cont. on $[0, \pi]$ and $-\cos x$ is an antiderivative of $\sin x$, we have,

$$\begin{aligned} \int_0^\pi \sin x &= [-\cos x]_0^\pi \\ &= -\cos \pi + \cos 0 \\ &= -(-1) + 1 = 2. // \end{aligned}$$

• The Fundamental Theorem of Calculus makes the connection bet$^{\underline{n}}$ integration and diff$^{\underline{n}}$.

If $f:[a,b]\to\mathbb{R}$ is an integrable fn which has an antiderivative F, then the fn $\mathcal{F}:[a,b]\to\mathbb{R}$ defined by $\mathcal{F}(x) := \int_a^x f(t)\,dt$ is also an antiderivative of f, since by the Fund. Thm we have,

$$\frac{d}{dx}\left(\int_a^b f(t)\,dt\right) = \frac{d}{dx}\left(F(x)-F(a)\right) = f(x) - 0 = f(x).\,//.$$