Orthogonal Bases & Gram-Schmidt:

Defⁿ: The vectors $q_1, q_2, \ldots, q_n$ are orthonormal if

$$q_i^T q_j = \begin{cases} 0 & \text{when } i \neq j \sim \text{orthogonality} \\ 1 & \text{when } i = j \sim \text{normalization.} \end{cases}$$

(A matrix with orthonormal columns will be called Q.)

e.g. - Standard basis.
- Rotation of this axes.

Theorem: If Q (sq. or rect.) has orthonormal colᵐ, then $Q^T Q = I$.

Proof:

$$\begin{pmatrix} - q_1^T - \\ - q_2^T - \\ \vdots \\ - q_n^T - \end{pmatrix} \begin{pmatrix} | & | & & | \\ q_1 & q_2 & \cdots & q_n \\ | & | & & | \end{pmatrix} = I_n.$$

Defⁿ: An orthogonal matrix is a sq. matrix with orthonormal columns.

Property: If Q is an orthogonal matrix, then $Q^T = Q^{-1}$.

Proof: $Q^T Q = I$ (Multiply row i of Q^T with colᵐ j of Q, to get $q_i^T q_j = 0$, except $q_i^T q_i = 1$.) //.

eg. Permutation matrices.
- Note that the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ reflects $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ ~ So reflection is also allowed.

Remark: Geometrically, an orthogonal Q is the product of a rotation + a reflection.

Theorem: Lengths remains unchanged when multiplied by Q.

Proof: We have $Q^T Q = I$, $\|Qx\|^2 = \|x\|^2$ because $(Qx)^T (Qx)$
$= x^T Q^T Q x = x^T x$. //.

Remark: Inner products and angles are also preserved.

(Since, $(Qx)^T (Qy) = x^T Q^T Q y = x^T y$.)

Q. Suppose we have an orthonormal basis $\{q_1, q_2, \dots, q_n\}$ of V. Then find the co-effs. of the eq^n, $b = x_1 q_1 + \dots + x_n q_n$.

Sol^n: - Multiply both sides by q_1^T which gives us, $q_1^T b = x_1 q_1^T q_1$
$\Rightarrow x_1 = q_1^T b$.

- Do this for other x_i's. //.

Thus, every vector b is equal to $(q_1^T b) q_1 + \dots + (q_n^T b) q_n$.

- So from the system $Qx = b$, we get the sol^n: $x = Q^T b$.

(This is easier than $Ax = b \Rightarrow x = A^{-1} b$.)

Gram-Schmidt Process: Let a, b, c be three vectors. If they are orthonormal, then to project a vector v onto a, we compute $(a^T v) a$. To project v onto the plane of a & b we compute $(a^T v) a + (b^T v) b$ and so on. What happens if they are not orthonormal? How to make them orthonormal?

Let's start with a, b, c again. We want now orthonormal vectors q_1, q_2 & q_3.

q_1 is easy, set $q_1 = \frac{a}{\|a\|}$.

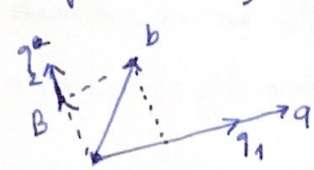
For q_2, we need it to be orthonormal to q_1. If b has any component in the direction of q_1 (same as a), that component will be subtracted.

Let $B = b - (q_1^T b) q_1$, then $q_2 = \frac{B}{\|B\|}$.

Similarly let $C = c - (q_1^T c) q_1 - (q_2^T c) q_2$, then $q_3 = \frac{C}{\|C\|}$.

This can now be extended for n vectors and this process is called the Gram-Schmidt process. (We subtract from every new vector its components in the directions that are already settled.)

(we remove the q_i component of b, ? and normalize a and b.) ←



eg: Let $a = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$, $b = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $c = \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$.

$$q_1 = \frac{a}{\|a\|} = \frac{a}{\sqrt{2}} := \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix}$$

$$B = b - (q_1^T b) q_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{pmatrix} = 1/2 \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

$$q_2 = \frac{B}{\|B\|}$$

$$C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad q_3 = \frac{C}{\|C\|}.$$

orthonormal basis: $Q = (q_1 \ q_2 \ q_3) = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \end{pmatrix}$ //.