Higher Order Derivatives:

Def^n: Given a diff. fn $f:[a,b]\to\mathbb{R}$ and $x_0\in(a,b)$ we define the derivative of order 2 of f at x_0 as the derivative of f' at x_0. i.e: $f''(x_0)=\lim\limits_{x\to x_0}\dfrac{f'(x)-f'(x_0)}{x-x_0}$.

Def^n: Given a fn $f:[a,b]\to\mathbb{R}$, $x_0\in(a,b)$, the derivative of order n for any $n\in\mathbb{N}$, denoted by $f^{(n)}(x_0)$ is defined as the derivative of $f^{(n-1)}$ at x_0.

- The set of all n-times diff. fns in (a,b) derivatives which are cont. is denoted by $C^n((a,b))$.

Def^n: Given a fn $f:[a,b]\to\mathbb{R}$, we say that f is cont. diff. if it is diff. on (a,b) and its derivative is cont.

eg: $f:(0,+\infty)\to\mathbb{R}$, $f(x)=\ln x$.

$f'(x)=1/x$, $f''(x)=(1/x)'=-1/x^2$, $f^{(3)}(x)=(-x^{-2})'=2/x^3$.

Algebra of diff. fns. of order n: Let $f,g:(a,b)\to\mathbb{R}$ be n-times diff. at $x_0\in(a,b)$. Then,

- The fn $h(x)=\alpha f(x)$, $\alpha\in\mathbb{R}$ is diff n times at x_0.
  $$h^{(n)}(x_0)=\alpha f^{(n)}(x_0).$$
- The fn $h(x)=f(x)+g(x)$, is diff n times at x_0.
- The fn $h(x)=f(x)g(x)$ is diff. n times at x_0.
  $$h^{(n)}(x_0)=\sum_{i=0}^{n}\binom{n}{i}f^{(i)}(x_0)\,g^{(n-i)}(x_0).$$
- If $g(x)\neq 0$ for all x near x_0 then $h(x)=\dfrac{f(x)}{g(x)}$ is n-times diff. at x_0.
- If $\tilde{g}$ is such that $\mathrm{Im}(f)\subset D(\tilde{g})$ and $\tilde{g}$ is n-times diff at $f(x_0)$, then $h(x):=\tilde{g}(f(x))$ is n times diff. at x_0.

<u>Proofs</u>: Uses induction on n //.

<u>eg</u>: Let $f: \mathbb{R} \to \mathbb{R}$ be trice diff and $h: \mathbb{R} \to \mathbb{R}$, $h(x) = e^{(f(x))^2}$.

$$h'(x) = 2f(x)f'(x)e^{(f(x))^2}$$

$$h''(x) = 2e^{(f(x))^2}\left(\left(2(f(x))^2 + 1\right)(f'(x))^2 + f(x)f''(x)\right). //$$

<u>Thm</u>: Let $f: (a,b) \to \mathbb{R}$ be trice diff. at $x_0 \in (a,b)$ and assume that x_0 is a stationary pt. Then,

(1) If $f''(x_0) > 0$, then x_0 is a local min$^{\underline{m}}$ pt.

(2) If $f''(x_0) < 0$, then x_0 is a local min$^{\underline{m}}$ pt.

[<u>Proof of (1)</u>: Let $f''(x_0) > 0$ at some stationary pt. x_0.

Then, $\lim \dfrac{f'(x) - f'(x_0)}{x - x_0} > 0$ (By def^n of $f''(x)$.)

$\Rightarrow$ There exists $\delta > 0$ s.t. $\dfrac{f'(x) - f'(x_0)}{x - x_0} > 0$ $\forall x \in (x_0 - \delta, x_0 + \delta)$, $x \neq x_0$.

For, $x_0 < x < x_0 + \delta$, we have $x - x_0 > 0$ so the above ineq gives us, $f'(x) > f'(x_0) = 0$.

Again, $x_0 - \delta < x < x_0 \Rightarrow x - x_0 < 0 \Rightarrow f'(x) < f'(x_0) = 0$.

Thus, since $f'(x) > 0$ $\forall x \in (x_0 - \delta, x_0)$, f is strictly increasing in $(x_0 - \delta, x_0)$, so that $f(x) > f'(x_0)$ $\forall x \in (x_0 - \delta, x_0)$.

Illy, $f(x) > f(x_0)$ $\forall x \in (x_0, x_0 + \delta)$.

$\Rightarrow f(x) > f(x_0)$ $\forall x \in (x_0 - \delta, x_0) \cup (x_0, x_0 + \delta)$.

$\Rightarrow f(x) \geq f(x_0)$ $\forall x \in (x_0 - \delta, x_0 + \delta)$. //]

<u>eg</u>: $f(x) = x^4 + x^2 + 1$, $f'(x) = 2x(2x^2 + 1)$, so $x = 0$ is the unique stationary pt. $f''(x) = 12x^2 + 2 \Rightarrow f''(0) = 2 > 0$, so $x_0 = 0$ is a local min$^{\underline{m}}$.

- In the case $f''(x_0) = 0$ we cannot deduce anything about the behaviour of f near x_0.

eg $x_0 = 0$ is a stationary pt. for $f(x) = x^4$, $g(x) = -x^4$ and $h(x) = x^3$. Every case 2nd derivative is 0 at origin. But $x_0 = 0$ is a min for f, max for g while neither max nor min for h.

- The result is only valid for stationary pts.

eg: $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$. $\Rightarrow f''(x) = 2 \quad \forall x \in \mathbb{R}$.

$x_0 = 0$ is a local max^{m} pt. only.

If we consider $x = 3$ which is not stationary then $f''(x) > 0$ but $x = 3$ is not a local min^{n} pt.

<u>De L'Hôpital rule of order n:</u> Let $f, g: [a,b] \to \mathbb{R}$ be cont. on $[a,b]$ and n-times diff. on (a,b). Let $x_0 \in (a,b)$ and $g^{(k)}(x) \neq 0$, $\forall x \in (a,b) \setminus \{x_0\}$ and $k = 0, 1, \ldots, n$, and

$$\lim_{x \to x_0} f^{(k)}(x) = \lim_{x \to x_0} g^{(k)}(x) = 0 \quad \forall k = 0, 1, \ldots, n-1.$$

If the limit of $f^{(n)}/g^{(n)}$ exists at x_0, then

$$\lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f^{(n)}(x)}{g^{(n)}(x)}.$$

<u>Proof</u>: Induction on n. //.

eg: $$\lim_{x \to 0} \frac{e^x - x - 1}{2x^2} = \lim_{x \to 0} \frac{e^x - 1}{4x} = \lim_{x \to 0} \frac{e^x}{4} = 1/4.$$

Taylor's Polynomial: Gives a way to approx. a fn near a given pt. by a poly. of order n, provided the fn is diff. at least n times. Use in computer algebra packages.

Defn: Let $f:(a,b)\to\mathbb{R}$ be diff. n times on (a,b) and $x_0, x \in (a,b)$. The fn,

$$P_n(x;x_0) := f(x_0) + f'(x_0)(x-x_0) + \frac{f^{(2)}(x_0)}{2!}(x-x_0)^2 + \dots$$
$$\dots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n,$$

is called the Taylor's polynomial of order n of $f(x)$ about the point x_0.

eg: $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^x$, $x_0=0$.

$$P_n(x;x_0) = 1 + 1(x-0) + \frac{1}{2!}(x-0)^2 + \dots + 1/n!\,(x-0)^n$$
$$= \sum_{k=0}^{n} 1/k!\; x^k.$$

eg: $f:\mathbb{R}\to\mathbb{R}$, $f(x)=e^{\alpha x}$, $\alpha\neq 0$, $x_0=0$.

$$P_n(x;x_0) = \sum_{k=0}^{n} \frac{1}{k!}(\alpha x)^k.$$

eg: $f:\mathbb{R}\to\mathbb{R}$, $f(x)=\sin x$, $x_0=0$.

$f'(x)=\cos x$, $f''(x)=-\sin x$, $f^{(3)}(x)=-\cos x$, $f^{(4)}(x)=\sin x$.

So, $f^{(4k+1)}(0)=1$, $f^{(4k+2)}(0)=0=f^{(4k+4)}(0)$, $f^{(4k+3)}(0)=-1$.

$$P_n(x;x_0) = \sum_{k=0}^{n-1}(-1)^k\frac{x^{2k+1}}{(2k+1)!}.$$

Theorem: Let $f, g : (a,b) \to \mathbb{R}$ be diff. n times on (a,b) and $x \in (a,b)$. Let $P_n^f(x; x_0)$, $P_n^g(x; x_0)$ be the Taylor poly. of order n about x_0 for f, g resp. Then,

(1) For any $\alpha \in \mathbb{R}$, the Taylor poly. of order n about x_0 for αf is $\alpha P_n^f(x; x_0)$.

(2) The Taylor poly of order n about x_0 for $f+g$ is given by $P_n^f(x; x_0) + P_n^g(x; x_0)$.

(3) Let $a<0$, $b>0$, the Taylor poly. of order n about $x_0 = 0$ for fg is given by all terms in the product of $P_n^f(x; 0) P_n^g(x; 0)$ up to degree n.

eg: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = \sin x$, $x_0 = 0$.

$$P_6^f(x; 0) = x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5.$$

$g(x) = x^2 + \sin x$, then, $P_6^g(x; 0) = x^2 + P_6^f(x; 0)$.

$h(x) = x^2 \sin x$, then $x^2 P_6^f(x; 0) = x^3 - \tfrac{1}{6}x^5 + \tfrac{1}{120}x^7$.

$$P_6^h(x; 0) = x^3 - \tfrac{1}{6}x^5.$$

Defn: The fn $R_n(x; x_0) = f(x) - P_n(x; x_0)$ is called the remainder of order n.

we write any fn. f as, $\underbrace{f(x) = P_n(x; x_0) + R_n(x; x_0)}_{\text{Taylor's expansion}}$.

Taylor's Theorem: Let $f : (a,b) \to \mathbb{R}$ be diff. $n+1$ times and $x_0, x \in (a,b)$. Then there exists a pt. ξ, s.t. $x < \xi < x_0$ such that
$$f(x) = P_n(x; x_0) + \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)^{n+1}.$$

- We can write the MVT as:

Given a cont. fn $f:[a,b]\to\mathbb{R}$, diff. on (a,b) and $x_0, x\in(a,b)$, then $\exists$ a pt. ξ bet^n x and x_0 s.t.

$$f(x)=f(x_0)+f'(\xi)(x-x_0).$$

(Taylor's theorem is a generalization for this.)

- $R_n(x;x_0)$ is a measure of how far away $P_n(x;x_0)$ is from being equal to $f(x)$. By Taylor's ~~sss~~ thm we have

$$R_n(x;x_0)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)^{n+1} \quad \text{for some } \xi \text{ bet}^n\ x \text{ and } x_0.$$

If we consider x_0 to be fixed then ξ depends only on x. Since ~~x~~ ξ is bet^n x and x_0, so $\xi\to x_0$ as $x\to x_0$. If $f^{(n+1)}$ is cont. then $f^{(n+1)}(\xi)\to f^{(n+1)}(x_0)$ as $x\to x_0$.

So, we have

$$\lim_{x\to x_0} R_n(x;x_0)=\frac{1}{(n+1)!}\lim_{x\to x_0} f^{(n+1)}(\xi)(x-x_0)^{n+1}$$

$$=\frac{1}{(n+1)!}f^{(n+1)}(x_0)\lim_{x\to x_0}(x-x_0)^{n+1}$$

$$=0.$$

$\boxed{\text{As } x\to x_0, \text{ the error}\to 0}$.

Corollary: Let $f:(a,b)\to\mathbb{R}$ be differentiable $(n+1)$ times and $x_0, x\in(a,b)$, then $|R_n(x;x_0)|\le \sup_{c\in[x,x_0]}|f^{(n+1)}(c)|\dfrac{|x-x_0|^{n+1}}{(n+1)!}$,

where, $\sup_{c\in[x,x_0]}|f^{(n+1)}(c)|$ means the sup. of the set of values of $|f^{(n+1)}(c)|\ \forall\, c\in[x,x_0]$.

- Consider $f(x) = e^{\alpha x}$, $\alpha \neq 0$, $x_0 = 0$.

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} (\alpha x)^k + R_n(x;0).$$

$$|R_n(1;0)| \leq \sup_{c \in [0,1]} |f^{(n+1)}(c)| \frac{|1-0|^{n+1}}{(n+1)!}$$

$$= \sup_{c \in [0,1]} |e^c| \frac{1}{(n+1)!} \leq \frac{3}{(n+1)!}.$$

Since $\frac{3}{(n+1)!} \to 0$ as $n \to \infty$, we see that $R_n(x;0) \to 0$ as $n \to \infty$.

So, $e^{\alpha x} = \sum_{k=0}^{\infty} \frac{1}{k!} (\alpha x)^k$.

- Consider $f(x) = \sin x$, $x_0 = 0$.

$$\sin x = \sum_{k=0}^{n-1} (-1)^k \frac{x^{2k+1}}{(2k+1)!} + R_{2n}(x;0).$$

$$|R_n(\alpha;0)| \leq \sup_{c \in [0,\alpha]} |f^{(n+1)}(c)| \frac{|\alpha - 0|^{n+1}}{(n+1)!} \leq \frac{\alpha^{n+1}}{(n+1)!}$$

$(\because |f^{(n+1)}(c)| \leq 1 \ \forall n)$.

$\frac{\alpha^{n+1}}{(n+1)!} \to 0$ as $n \to +\infty$, so $R_n(x;0) \to 0$ as $n \to \infty$.

$$\therefore \sin x = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}.$$