Properties of Infinite Series:

Recall, $\sum_{k=1}^{\infty} a_k = A$ means that $\lim s_n = A$.

Theorem: If $\sum_{k=1}^{\infty} a_k = A$ and $\sum_{k=1}^{\infty} b_k = B$, then

(i) $\sum_{k=1}^{\infty} c a_k = cA \ \forall c \in \mathbb{R}$,

(ii) $\sum_{k=1}^{\infty} (a_k + b_k) = A + B$.

Proof: Apply algebra of limits type arguments.

Cauchy Criterion for Series: The series $\sum_{k=1}^{\infty} a_k$ converges iff given $\varepsilon > 0$, $\exists N \in \mathbb{N}$ s.t. whenever $n > m \geq N$, we have

$$|a_{m+1} + a_{m+2} + \cdots + a_n| < \varepsilon.$$

Proof: $|s_n - s_m| = |a_{m+1} + \cdots + a_n| < \varepsilon$ by Cauchy Criterion for seq$^{\underline{n}}$s.

Theorem: If the series $\sum_{k=1}^{\infty} a_k$ converges, then $(a_k) \to 0$.

Proof: Take $n = m+1$ in the Cauchy criterion for Series.

- The converse is not true (cf. Harmonic Series).

Comparision Test: Let (a_k) and (b_k) are seq$^{\underline{n}}$s satisfying $0 \leq a_k \leq b_k \ \forall k \in \mathbb{N}$.

(i) If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.

(ii) If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} b_k$ diverges.

Proof: Use Cauchy Criterion for Series and

$$|a_{m+1} + \cdots + a_n| \leq |b_{m+1} + \cdots + b_n|. \,/\!/$$

<u>Absolute Convergence Test</u>: If the series $\sum_{n=1}^{\infty} |a_n|$ converges, then the series $\sum_{n=1}^{\infty} a_n$ converges.

<u>Proof</u>: Since $\sum_{n=1}^{\infty} |a_n|$ converges, given an $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t.

$|a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \varepsilon$, $\forall\, n > m \geq N$, (by the Cauchy criterion for Series) $|a_{m+1} + \dots + a_n| \leq |a_{m+1}| + \dots + |a_n| = \varepsilon$ guarantees that $\sum_{n=1}^{\infty} a_n$ converges. //.

- The converse is false: $1 - 1/2 + 1/3 - 1/4 + \dots$ converges.

<u>Alternating Series Test</u>: Let (a_n) be a seq$^{\underline{n}}$ satisfying $a_1 \geq a_2 \geq a_3 \geq \dots \geq a_n \geq a_{n+1} \geq \dots$ and $(a_n) \to 0$. Then, the alternating series $\sum_{n=1}^{\infty} (-1)^{n+1} a_n$ converges.

<u>Proof</u>: Proving this is same as showing $s_n = a_1 - a_2 + \dots \pm a_n$ converges. Just show (s_n) is Cauchy. //.

<u>Def^n</u>: If $\sum_{n=1}^{\infty} |a_n|$ converges, then we say that $\sum_{n=1}^{\infty} a_n$ <u>converges absolutely</u>.

If $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ doesn't then we say that $\sum_{n=1}^{\infty} a_n$ <u>converges conditionally</u>.

<u>eg</u>: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges conditionally, whereas $\sum_{n=1}^{\infty} 1/2^n$ converges absolutely.

<u>Def^n</u>: Let $\sum_{k=1}^{\infty} a_k$ be a series. A series $\sum_{k=1}^{\infty} b_k$ is called a <u>rearrangement</u> of $\sum_{k=1}^{\infty} a_k$ if $\exists$ a 1-1, onto fn $f: \mathbb{N} \to \mathbb{N}$ s.t. $b_{f(k)} = a_k\ \forall\, k \in \mathbb{N}$.

<u>eg</u>: $1 + 1/3 - 1/2 + 1/5 + 1/7 - 1/4 + \dots$ is a rearrangement of $1 - 1/2 + 1/3 - 1/4 + \dots$.

Theorem: If a series converges absolutely, then any rearrangement of the series converges to the same limit.

Proof: Let $\sum_{k=1}^{\infty} a_k$ converge absolutely to A, and let $\sum_{k=1}^{\infty} b_k$ be a rearrangement of $\sum_{k=1}^{\infty} a_k$.

Let $s_k = a_1 + a_2 + \dots + a_k$ and $t_k = b_1 + b_2 + \dots + b_k$.

We want to show $(t_k) \to A$ as well.

Let $\varepsilon > 0$. Choose N_1 s.t. $\forall\, n \geq N_1$ we have $|s_n - A| < \varepsilon/2$.

Choose N_2 s.t. $\forall\, n > m \geq N_2$ $|a_{m+1}| + |a_{m+2}| + \dots + |a_n| < \varepsilon/2$.

Take $N = \max\{N_1, N_2\}$ and choose $M = \max\{f(k) : 1 \leq k \leq N\}$.

Now, if $m \geq M$, then $(t_m - s_N)$ consists of a finite set of terms. Our N_2 gives us $|t_m - s_N| < \varepsilon/2$.

So, $|t_m - A| \leq |t_m - s_N| + |s_N - A| < \varepsilon/2 + \varepsilon/2 = \varepsilon$. //