Integration:

Theorem: (a) Let $f:[a,b]\to\mathbb{R}$ ~~and g~~ be an integrable fn which has an antiderivative F, and let $\alpha\in\mathbb{R}$. Then the fn $h:[a,b]\to\mathbb{R}$ defined by $h(x)=\alpha f(x)$ is integrable on $[a,b]$ and $\int_a^b h(x)\,dx=\alpha\int_a^b f(x)\,dx$.

(b) Let $f,g\to\mathbb{R}$ be integrable fns which have antiderivatives F, G resp. Then the fn $h:[a,b]\to\mathbb{R}$ defined by $h(x)=f(x)+g(x)$ is integrable on $[a,b]$ and $\int_a^b h(x)\,dx=\int_a^b f(x)\,dx+\int_a^b g(x)\,dx$.

(c) (Integration by parts) Let $f,g:[a,b]\to\mathbb{R}$ be fns which are continuously diff. on (a,b) i.e. f', g' are cont. on (a,b). Then

$$\int_a^b f(x)g'(x)\,dx=\Big[f(x)g(x)\Big]_a^b-\int_a^b f'(x)g(x)\,dx.$$

(d) (Integration by substitution) Let $I\subset\mathbb{R}$ be an interval, $f:I\to\mathbb{R}$ a cont. fn with antiderivative F, and $\psi:[a,b]\to I$ a continuously diff. fn. Then, with $u=\psi(x)$, we have

$$\int_a^b f(\psi(x))\psi'(x)\,dx=\int_{\psi(a)}^{\psi(b)} f(u)\,du.$$

Proof: (a) & (b) follows from properties for antiderivatives and the Fund. Thm. Calculus.

(c) Note fg is diff. over (a,b) as f & g are diff., also

$$(f(x)g(x))'=\underbrace{f'(x)g(x)+f(x)g'(x)}$$

↳ prod. of cont. fns so cont. and hence integrable.

$\Rightarrow$ Same is true for LHS.

Now we integrate RHS over $[a,b]$ and use (b) to get,

$$\int_a^b (f'(x)g(x)+f(x)g'(x))\,dx = \int_a^b f'(x)g(x)\,dx + \int_a^b f(x)g'(x)\,dx.$$

By the fund. thm. of calculus, the integral of the LHS is,

$$\int_a^b (f(x)g(x))'\,dx = f(b)g(b) - f(a)g(a) = [f(x)g(x)]_a^b.$$

The result now follows.//

(d) Since F is ~~the~~ an antiderivative of f, then $F\circ\psi$ is an antiderivative of $(f\circ\psi)\psi'$, that is $(F(\psi(x)))' = F'(\psi(x))\psi'(x)$

$$= \underbrace{f(\psi(x))\,\psi'(x)}_{\text{integrable on } [a,b]}.$$

By Fund. Thm, we get $\int_a^b f(\psi(x))\psi'(x)\,dx = [F(\psi(x))]_a^b$.

Illy, since f is int. on $[\psi(a),\psi(b)]\subset I$ and has an antiderivative F, by the fund. thm we get $\int_{\psi(a)}^{\psi(b)} f(u)\,du = [F(u)]_{\psi(a)}^{\psi(b)}$.

The proof is complete.//

eg: $\int_0^2 te^t\,dt = [te^t]_0^2 - \int_0^2 1.e^t\,dt = 2e^2 - 0 - \int_0^2 e^t\,dt$

$$= 2e^2 - [e^t]_0^2 = 2e^2 - e^2 + e^0$$

$$= e^2 + 1.//$$

eg: $\int_5^{3\pi} e^x \sin x\,dx = [e^x \sin x]_5^{3\pi} - \int_5^{3\pi} e^x \cos x\,dx$

$$= e^{3\pi}\sin(3\pi) - e^5 \sin 5 - \int_5^{3\pi} e^x\cos x\,dx \quad \text{— ①}$$

$\int_5^{3\pi} e^x\cos x\,dx = [e^x \cos x]_5^{3\pi} - \int_5^{3\pi} e^x(-\sin x)\,dx$

$$= e^{3\pi}\cos(3\pi) - e^5\cos 5 + \int_5^{3\pi} e^x \sin x\,dx \quad \text{— ②}$$

① & ② $\Rightarrow$ $2\int_5^{3\pi} e^x \sin x\,dx = e^{3\pi} + e^5(\cos 5 - \sin 5)$.//

eg: $\displaystyle\int_0^{1/2} \frac{x^3}{\sqrt{1-x^2}}\,dx = \int_0^{1/2} \frac{x^2}{\sqrt{1-x^2}}\,x\,dx$

$$= -\int_{u(0)}^{u(1/2)} \frac{1-u^2}{u}\,u\,dx \qquad \left[u = \sqrt{1-x^2}\right]$$

$$= -\int_1^{\sqrt{3}/2} (1-u^2)\,du = \left[-u + \frac{u^3}{3}\right]_1^{\sqrt{3}/2} \quad //.$$

Note: $\displaystyle\int_a^b f(u(x))\,dx \neq \left[\frac{1}{u'(x)} F(u(x))\right]_a^b$.

More examples:

(1) $\displaystyle\int_a^t \ln x\,dx = \int_a^t \ln x \cdot 1\,dx = \int_a^t \ln x \cdot (x)'\,dx = \left[\ln x \cdot x\right]_a^t - \int_a^t \frac{1}{x}\,x\,dx.$

$$= t\ln t - a\ln a - t + a \quad //.$$

(2) $\displaystyle\int_a^y \cot x\,dx = \int_a^y \frac{1}{\sin x}\cos x\,dx = \int_{u(a)}^{u(y)} \frac{1}{u}\,du \qquad [u = \sin x]$

$$= \left[\ln|u|\right]_{\sin a}^{\sin y} = \ln|\sin y| - \ln|\sin a|.$$

(3) $\displaystyle\int_a^t \frac{x^2}{64+x^6}\,dx = \frac{1}{3}\int_a^t \frac{1}{64+x^6}\,3x^2\,dx = \frac{1}{3}\int_{u(a)}^{u(t)} \frac{1}{8^2+u^2}\,du$

$$[u = x^3].$$

$$= \frac{1}{3}\left[\frac{1}{8}\arctan\left(\frac{u}{8}\right)\right]_{a^3}^{t^3} \quad //.$$

(4) $\displaystyle\int_a^y \frac{1}{5\cosh x - 3\sinh x}\,dx = \int_a^y \frac{1}{\frac{5}{2}(e^x+e^{-x}) - \frac{3}{2}(e^x-e^{-x})}\,dx$

$$= \int_a^y \frac{e^x}{e^{2x}+4} = \int_{u(a)}^{u(y)} \frac{1}{u^2+2^2}\,du \qquad [u = e^x]$$

$$= \frac{1}{2}\left[\arctan\left(\frac{u}{2}\right)\right]_{e^a}^{e^y} \quad //.$$

(5) $$\int_a^t \sin^5 x \cos^6 x\, dx = \int_a^t \sin^4 x \cos^6 x \sin x\, dx$$

$$= \int_a^t (1-\cos^2 x)^2 \cos^6 x \sin x\, dx$$

$$= -\int_{u(a)}^{u(t)} (1-u^2)^2 u^6\, du \quad [u = \cos x]$$

(Just expand and integrate.) //.

(6) $$\int_a^t \frac{x+3}{x^2+2x+10}dx = \int_a^t \frac{x+3}{(x+1)^2+9}dx = \int_{u(a)}^{u(t)} \frac{u+2}{u^2+9}du \quad (u = x+1)$$

$$= \int_{a+1}^{t+1} \frac{u}{u^2+9}dx + 2\int_{a+1}^{t+1} \frac{1}{u^2+9}du$$

$$= \frac{1}{2}\left[\ln(u^2+9)\right]_{a+1}^{t+1} + 2\left[\tfrac{1}{3}\arctan\frac{u}{3}\right]_{a+1}^{t+1} \quad //.$$

(7) $$\int_a^t \frac{3}{2x^2-3x-2}dx = \frac{3}{5}\int_a^t \left(\frac{-2}{2x+1} + \frac{1}{x-2}\right)dx. \quad //.$$

(8) $$\int_a^t \frac{x^2-5}{x^4-6x^2-7x-6}dx = \int_a^t \frac{x^2-5}{(x-3)(x+2)(x^2+x+1)}dx$$

$$= \int_a^t \left[\frac{4}{65(x-3)} + \frac{1}{15(x+2)} + \frac{-5x+32}{39(x^2+x+1)}\right]dx \quad //$$

(9) $$\int_a^t \frac{2x^4-3x^3+39x^2-28x+190}{x^5-x^4+18x^3-18x^2+81x-81}dx$$

$$= \int_a^t \frac{2x^4-3x^3+39x^2-28x+190}{(x-1)(x^2+9)^2}dx$$

$$= \int_a^t \left[\frac{2}{x-1} - \frac{3}{x^2+9} - \frac{1}{(x^2+9)^2}\right]dx. \quad //.$$

(10) $$\int_a^t \frac{x^4+3x^3-1}{x^3-x^2+2x-2}\,dx = \int_a^t \left[(x+4) + \frac{2x^2-7x+8}{(x^2+2)(x-1)}\right] dx$$

$$= \int_a^t \left(x+4 + \frac{x}{x^2+2} - \frac{6}{x^2+2} + \frac{1}{x-1}\right) dx \,//.$$

Some more properties of integrals:

Additivity: Let $f:[a,b]\to\mathbb{R}$ and $c\in[a,b]$, then f is integrable on $[a,b]$ iff f is integrable on $[a,c]$ and $[c,b]$, and $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.

eg: $$\int_0^2 |x^2-3|\,dx = \int_0^{\sqrt{3}} (-x^2+3)\,dx + \int_{\sqrt{3}}^2 (x^2-3)\,dx \,//.$$

- Let f be integrable on $[a,b]$ and let g be a fn which differs from f at a finite no. of pts on $[a,b]$. Then g is also integrable on $[a,b]$ and $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$.

- Let $f_1:[a,b]\to\mathbb{R}$, $f_2:[b,c]\to\mathbb{R}$ be cont.

  (a) Define the piecewise cont fn $f:[a,c]\to\mathbb{R}$ by,
  $$f(x) = \begin{cases} f_1(x) & \text{if } x\in[a,b) \\ f_2(x) & \text{if } x\in[b,c]. \end{cases}$$
  Then f is int. and $\int_a^c f(x)\,dx = \int_a^b f_1(x)\,dx + \int_b^c f_2(x)\,dx$.

  (b) Define the piecewise cont fn $f:[a,c]\to\mathbb{R}$ by
  $$f(x) = \begin{cases} f_1(x), & \text{if } x\in[a,b] \\ f_2(x), & \text{if } x\in(b,c] \end{cases}$$
  Then f is int. and $\int_a^c f(x)\,dx = \int_a^b f_1(x)\,dx + \int_b^c f_2(x)\,dx$.

<u>Corollary</u>: Let $f : [a,b] \to \mathbb{R}$ be a bdd piecewise cont^n fn.
Then f is integrable on $[a,b]$.

- Bdd is essential as a piecewise cont. fn could be unbdd eg: $f : [0,1] \to \mathbb{R}$ defined by $f(x) = \begin{cases} 1/x, & x > 0 \\ 0, & x = 0. \end{cases}$

<u>eg</u>: $f(x) = \begin{cases} x^2 & , -2 \le x \le -1 \\ x^5 - x^2 + 3 & , -1 \le x < 1 \\ -x^3 + 3x + 1 & , 1 \le x < 2 \\ -4x^2 + 5x + 5 & , 2 \le x \le 5/2. \end{cases}$

$$\int_{-2}^{2} f(x)\,dx = \int_{-2}^{1} x^2\,dx + \int_{-1}^{1} (x^5 - x^2 + 3)\,dx + \int_{1}^{2} (-x^3 + 3x + 1)\,dx$$ //.

In fact, the following facts have been assumed so far by us,
- All cont^n fns have an antiderivative.
- All cont^n fns are integrable.

The proofs are NOT required for this course.

<u>Improper Integrals</u>:

<u>Def^n</u>: (a) Let $\alpha \in \mathbb{R}$, $f : [\alpha, +\infty) \to \mathbb{R}$ be integrable on $[\alpha, b]$ $\forall$ $b > \alpha$. Then the <u>improper integral of f over $[\alpha, +\infty)$</u> is defined by $\int_{\alpha}^{+\infty} f(x)\,dx := \lim_{b \to +\infty} \int_{\alpha}^{b} f(x)\,dx$.

The integral is said to <u>converge</u> if the limit exists, otherwise it <u>diverges</u>.

(b) Illy, for $f : (-\infty, b] \to \mathbb{R}$ int^n on $[a,b]$ for all $b > 0$, we define the <u>improper integral of f over $(-\infty, b]$</u>

by $\displaystyle\int_{-\infty}^{b} f(x)\,dx := \lim_{a \to -\infty} \int_{a}^{b} f(x)\,dx.$

(c) If $\cancel{\mathbb{R}}$ $f : \mathbb{R} \to \mathbb{R}$ is integrable on $[a,b]$ $\forall\, a,b \in \mathbb{R}$, $b > a$, we define the <u>improper integral of f over $\mathbb{R}$</u> by

$$\int_{-\infty}^{+\infty} f(x)\,dx := \int_{-\infty}^{c} f(x)\,dx + \int_{c}^{+\infty} f(x)\,dx \quad \text{for any } \cancel{\forall}\; c \in \mathbb{R}.$$

<u>eg</u>: $\displaystyle\int_0^\infty \frac{1}{x^2+1}\,dx = \lim_{b \to \infty} \int_0^b \frac{1}{1+x^2}\,dx = \lim_{b \to \infty} \left[\arctan x\right]_0^b$

$\displaystyle = \lim_{b \to \infty} (\arctan b) = \pi/2.$ //.

<u>Theorem:</u> (1.) Let $f, g : [a, +\infty) \to \mathbb{R}$ be int. on $[a,b]$ $\forall\, b > a$.

(a) Let $h : [a, +\infty) \to \mathbb{R}$ be defined by $h(x) = \alpha f(x)$, $\alpha \in \mathbb{R}$.

Then $\displaystyle\int_a^\infty h(x)\,dx = \alpha \int_a^\infty f(x)\,dx.$

(b) Let $h : [a, +\infty) \to \mathbb{R}$ be defined by $h(x) = f(x) + g(x)$. Then,

$$\int_a^\infty h(x)\,dx = \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx.$$

(c) For $c > a$, we have $\displaystyle\int_a^\infty f(x)\,dx = \int_a^c f(x)\,dx + \int_c^\infty f(x)\,dx.$

(2) Similar results hold for $f, g : (-\infty, b] \to \mathbb{R}$ which are int. on $[a,b]$ $\forall\, b > a$.

(3) Similar results hold for $f, g : \mathbb{R} \to \mathbb{R}$ int. on $[a,b]$ $\forall\, a, b \in \mathbb{R}$, $b > a$.

<u>Def^n:</u> (a) Suppose $f : [a, b) \to \mathbb{R}$ is integrable over the interval $[a,c]$ $\forall\, c \in (a,b)$ but has an infinite discontinuity at b, then the improper integral of f over $[a,b]$ is defined by

$$\int_a^b f(x)\,dx := \lim_{c \to b^-} \int_a^c f(x)\,dx.$$

The integral is said to <u>converge</u> if the limit exists, and <u>diverge</u> otherwise.

(b) Illy, if $f:(a,b] \to \mathbb{R}$ is int. over $[c,b]$ $\forall\, c \in (a,b]$, but has an infinite discontinuity at a, we define

$$\int_a^b f(x)\,dx := \lim_{c \to a+} \int_c^b f(x)\,dx.$$

(c) If f is cont. on $[a,b]$, except for an infinite discon--tinuity at a pt $c \in (a,b)$ then the improper integral of f over $[a,b]$ is defined by $\int_a^b f(x)\,dx := \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$, where the two integrals on the R.H.S. are improper integrals.

eg: $$\int_0^1 \frac{1}{\sqrt{1-x}}\,dx = \lim_{c \to 1-} \int_0^c \frac{1}{\sqrt{1-x}}\,dx = \lim_{c \to 1-} \left[-2\sqrt{1-x}\right]_0^c$$
$$= \lim_{c \to 1-} \left(-2\sqrt{1-c} + 2\right) = 2. //$$

eg: $$\int_1^4 \frac{1}{(x-2)^{2/3}}\,dx = \int_1^2 \frac{1}{(x-2)^{2/3}}\,dx + \int_2^4 \frac{1}{(x-2)^{\cancel{4}\,2/3}}\,dx.$$

$$\int_1^2 \frac{1}{(x-2)^{2/3}}\,dx = \lim_{c \to 2-} \int_1^c \frac{1}{(x-2)^{2/3}}\,dx = \lim_{c \to 2-} \left[3(x-2)^{1/3}\right]_1^c$$
$$= \lim_{c \to 2-} \left(3(c-2)^{1/3} - 3(-1)^{1/3}\right) = 3.$$

Illy, $$\int_2^4 \frac{1}{(x-2)^{2/3}}\,dx = 3\sqrt[3]{2}. //$$

<u>Remark:</u> Properties of definite integrals (multiplication by scalar, sum & additivity) carry forward to these improper integrals as well.
