

Vector Spaces:

We are already familiar with some examples of vector spaces.

$\mathbb{R} \sim$ set of real numbers

$\mathbb{R}^2 \sim$ x-y plane

$\mathbb{R}^3 \sim$ 3-D space

$\mathbb{R}^n \sim$ all column vectors with n components.

We have two operations: addition of two vectors
& multiplication by a scalar

For us, scalars will always be real numbers.

In addition, there is also a "zero vector" and an "inverse vector".

Defⁿ: A real vector space is a set of vectors together with rules for vector addition and scalar multiplication.

It satisfies the following properties:

(Here x, y are vectors and c, c_1, c_2 are scalars)

(1) $x + y = y + x$

(2) $x + (y + z) = (x + y) + z$

(3) There exist a unique element '0' s.t. $x + 0 = x$.

(4) There exist a unique element '-x' for every vector

x s.t. $-x + x = 0$.

(5) $1 \cdot x = x$

$$(6) (c_1 c_2)x = c_1(c_2 x).$$

$$(7) c(x+y) = cx + cy.$$

$$(8) (c_1 + c_2)x = c_1 x + c_2 x$$

Examples: (1) The space of all $m \times n$ matrices.

(2) The space of functions $f(x)$, defined on a fixed interval, say $0 \leq x \leq 1$.

(3) Infinite dimensional space \mathbb{R}^∞ with infinitely many components.
Caution: we will explain this later

(4) Any plane in \mathbb{R}^3 passing through $(0,0,0)$.

This example suggests/motivates the concept of a subspace.

Defⁿ: A subspace of a v.s. is a nonempty subset that satisfies the requirements of a v.s.

- Linear combinations stay in the subspace.
- The "zero vector" belongs to any subspace.

Examples: (1) The "zero vector" is the smallest subspace.

(2) The whole space is the largest subspace.

(3) Four types of subspaces of \mathbb{R}^3 :

- \mathbb{R}^3 itself
- Any plane through $(0,0,0)$
- Any line through $(0,0,0)$

- The zero vector: $(0,0,0)$.

(4) The set of lower triangular matrices.

Non-examples: (1) All vectors in \mathbb{R}^2 whose components are positive or zero.

(2) Including negative components also does not result in a subspace.

Our motivation/goal is to understand equations of the type $Ax = b$ more deeply.

To that end we will next discuss two important types of v.s.: - Column space & Null space.

