<u>Column Space</u>: The column space contains all linear combinations of the columns of the matrix A.

- Denoted by $C(A)$.
- $C(A)$ is a subspace of $\mathbb{R}^m$ (for appropriate m).

<u>eg</u>: $m=3$, $n=2$ unknowns

$$\begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \underbrace{\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}}_{\text{combination of all columns.}}$$

The system $Ax=b$ is solvable iff the vector b can be expressed as a combination of the col$^{\underline{m}}$ of A.

Then b is the column space.

- All attainable r.h.s. b are all combinations of the columns of A.

- There are at least three possibilities.
  - $u=1, v=0 \Rightarrow b$ is the 1^{st} col$^{\underline{m}}$
  - $u=0, v=1 \Rightarrow b$ ... 2^{nd} ...
  - $u=0, v=0 \Rightarrow b=0$.

<u>Geometrically</u>, $Ax=b$ can be solved iff b lies in the <u>plane</u> spanned by the col$^{\underline{m}}$ vectors.

If b lies off this plane then $Ax=b$ has no sol$^{\underline{n}}$.

<u>Check</u>: $C(A)$ is actually a subspace of $\mathbb{R}^m$.

Nullspace: The solⁿ of $Ax=0$ also form a v.s. This v.s. is called the <u>nullspace</u>.

Defⁿ: The <u>nullspace</u> of a matrix A, denoted by $N(A)$ consists of all vectors x s.t. $Ax=0$.

Check: $N(A)$ is a subspace of $\mathbb{R}^n$.

eg: $\begin{pmatrix} 1 & 0 \\ 5 & 4 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $N(A) = (0,0)$ $[u=0, v=0]$.

Goal: For any system $Ax=b$ we want to find $C(A)$ & $N(A)$.

All attainable r.h.s. b ⟵ $C(A)$

$N(A)$ ⟶ All solⁿˢ of $Ax=0$.

- The vectors b are in the solⁿ space.
- The vectors x are in the null space.

Solving $Ax=b$ & $Ax=0$:

There was one solⁿ of $Ax=b$ and that was

$$x = A^{-1}b$$

(we found this via elimination, not by finding A^{-1})

If we have a rectangular matrix then that may not have a full set of pivots.

Now we would like to reduce such a matrix to one that we can work with.

For an invertible matrix, the null space contains only $x=0$. $(A^{-1} \cdot Ax = 0)$.

The col$^{\underline{n}}$ space is the whole space ($Ax=b$ has a sol^n for every b.)

<u>Question:</u> What happens when

- null space has more than the zero vector?
- col$^{\underline{n}}$ space has less than all vectors?

<u>Answer:</u> - Any $x_n \in N(A)$ can be added to a particular sol$^{\underline{n}}$ x_p. The sol$^{\underline{n}}$ to all linear eq$^{\underline{n}}$ have the form $x = x_n + x_p$.

$$(Ax_p = b,\ Ax_n = 0 \Rightarrow A(x_p + x_n) = b)$$

- When $C(A)$ doesn't contain every b in $\mathbb{R}^m$, we need the conditions on b that make $Ax=b$ solvable.

<u>Recall Echelon matrix:</u>

$$\begin{pmatrix} \bullet & \times & \times & \times & \times \\ 0 & \bullet & \times & \times & \times \\ 0 & 0 & \bullet & \times & \times \\ 0 & 0 & 0 & \bullet & \times \\ 0 & 0 & 0 & 0 & \bullet \end{pmatrix}$$

<u>Echelon form:</u> Echelon matrix U has a "staircase pattern"

$$\begin{pmatrix} \bullet & \times & \times & \times & \times & \times & \times \\ 0 & \bullet & \times & \times & \times & \times & \times \\ 0 & 0 & 0 & 0 & \bullet & \times & \times \\ 0 & 0 & 0 & 0 & 0 & \bullet & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow$$ reduced so that $\bullet$ become 1 and everything above 1 is 0.

Row reduced form: $R = \begin{pmatrix} 1 & 0 & \times & \times & 0 & 0 & \times \\ 0 & 1 & \times & \times & 0 & 0 & \times \\ 0 & 0 & 0 & 0 & 1 & 0 & \times \\ 0 & 0 & 0 & 0 & 0 & 1 & \times \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

- When A is invertible, $R = I_n$.
- $Rx = 0$ has the same sol^n as $Ux = 0$ which has the same sol^n as $Ax = 0$.

We want to read off the sol^n of $Rx = 0$.

For eg. say $U = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$R = \begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ $(R_1 - 3R_2)$

$Rx = \begin{pmatrix} 1 & 3 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \\ w \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

pivot col^m

pivot variables: u, w
free variables: v, y

To find general sol^n for $Rx = 0$ (eq. $Ax = 0$) assign arbitrary values to free variables.

So, $\left.\begin{aligned} u + 3v - y &= 0 \\ w + y &= 0 \end{aligned}\right\} \Rightarrow \begin{aligned} u &= -3v + y \\ w &= -y \end{aligned}$

The complete sol^n is a combination of two special sol^{ne}:

$$x = \begin{pmatrix} -3v+y \\ v \\ -y \\ y \end{pmatrix} = v\begin{pmatrix} -3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + y\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

Special sol^{n}s: $(-3,1,0,0)$ for $v=1$, $y=0$

$(1,0,-1,1)$ for $v=0$, $y=1$

All other sol^n are linear combinations of these two.

This forms the nullspace i.e. sol^{n}s of $Ax=0$.

Theorem: If a matrix has more colums than rows, $n>m$.

There must be at least $n-m$ free variables.

(Since m rows can have at most m pivots.)

Corollary: If $Ax=0$ has more unknowns than eq^{ns} $(n>m)$, then it has at least one special sol^n. In particular, there are more sol^{n}s than the trivial sol^n.

Let's now look at the case when $b \neq 0$.

Let, $A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 7 \\ -1 & -3 & 3 & 4 \end{pmatrix}$, This gives $U = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$.

Set $b = (b_1, b_2, b_3)$ to get,

$$Ux = c \text{ where } c = (b_1,\ b_2 - 2b_1,\ b_3 - 2b_2 + 5b_1)$$
$$= L^{-1}b \text{ (from previous lecture)}$$

Recall, $Ax=b$ can be solved iff b lies in $C(A)$.

Columns of A: $\begin{pmatrix}1\\2\\-1\end{pmatrix}, \begin{pmatrix}3\\6\\-3\end{pmatrix}, \begin{pmatrix}3\\9\\3\end{pmatrix}$ & $\begin{pmatrix}2\\7\\4\end{pmatrix}$

thrice of col 1. $\qquad$ $C_3 - C_1$

no pivots (check U)

$C(A)$ is the plane generated by C_1 and C_3.

Equivalently, it is also the plane of all vectors b satisfying

$b_3 - 2b_2 + 5b_1 = 0$ (see c last component).

Every col^{m} actually satisfies this, so it is forced on b.

Choose $b = (1,5,5)$ s.t. $b_3 - 2b_2 + 5b_1 = 0$

$Ux = c$ has sol^{ns}: $w = 1 - y$

$u = -2 - 3v + y$ where v & y are free variables.

Complete sol^{n}s: $x = x_p + x_n$

$$x = \begin{pmatrix}u\\v\\w\\y\end{pmatrix} = \underbrace{\begin{pmatrix}-2\\0\\1\\0\end{pmatrix}} + v\begin{pmatrix}-3\\1\\0\\0\end{pmatrix} + y\begin{pmatrix}1\\0\\-1\\1\end{pmatrix}$$

x_p, a particular sol^{n} of $Ax=b$.
(Obtained from $v=0, y=0$)

Summary: Let's say elimination reduces $Ax=b$ to $Ux=c$ and $Rx=d$, with r pivots rows and r pivot col^{m}.

The rank of these matrices is then r.

The last $m-r$ rows of U & R are 0, so there is a sol^{n} only if the last $m-r$ entries of c and d are also 0.

The complete sol^{n} is $x = x_p + x_n$.

x_n ↳ $N(A)$ combination of $n-r$ special sol^{n}s.

↓ obtained by setting one free variable equal to 1

x_p ↓ all free variables set to 0.
