Linear independence, Basis & Dimension:

- Recall, rank was introduced as the no. of pivots in the elimination process.
- There is actually a better description for rank.

Take $A = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 2 & 6 & 9 & 5 \\ -1 & -3 & 3 & 0 \end{pmatrix}$ ~ in reality there are only three columns which are independent of each other.

$2R_1$

$3C_1$

~ NOT all rows are independent.

The rank counts the no. of independent rows in the matrix.

Defⁿ: Suppose $c_1v_1 + c_2v_2 + \cdots + c_nv_n = 0$ only happens when $c_1 = c_2 = \cdots = c_n = 0$ for scalars c_i $(i = 1, 2, \ldots, n)$ and vectors v_i $(i = 1, 2, \ldots, n)$. Then, the vectors $v_1, v_2, \ldots, v_n$ are said to be linearly independent.

If any $c_i \neq 0$ then the vectors are linearly dependent.

eg: In $\mathbb{R}^3$, two vectors are independent if they do not lie on the same line.

three vectors are - - - - - - - - - . . lie on the same plane.

(What about four vectors?)

When are the colᵐˢ of A independent?

— When $N(A) = \{0\}$.

- The non-zero rows of any echelon matrix U are indep.

- If we pick the pivot colⁿ they are also independent.

Defⁿ: If a v.s. V consists of all linear combinations of $v_1, v_2, \dots, v_k$, then these vectors span the space.

i.e. Every $v \in V$ is a combination of these v_i's.

Defⁿ: A basis for a v.s. V is a set of vectors which are linearly independent and span the space V.

eg: The co-ordinate vectors $e_1, e_2, \dots, e_n$ span $\mathbb{R}^n$.

Is this a basis for $\mathbb{R}^n$?

Question: Is a basis unique?

Thm: For a vector $v \in V$, there is only one way to write it as a combination of the basis vectors.

Proof: Suppose there were two such ways,

Say $v = a_1 v_1 + \dots + a_n v_n$ and

$v = b_1 v_1 + \dots + b_n v_n$, where $\{v_1, \dots, v_n\}$ is a basis.

Then this immediately gives us, $(a_1 - b_1)v_1 + \dots + (a_n - b_n)v_n = 0$

And since the basis vectors are independent so, each $a_i - b_i = 0$.

$\Rightarrow a_i = b_i$. //

A v.s. has infinitely many different bases.

e.g: Whenever A is invertible, its colⁿ are independent and they are a basis for $\mathbb{R}^n$.

For a echelon matrix say $U = \begin{pmatrix} 1 & 3 & 3 & 2 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ an easy choice for a basis for $C(U)$ are the pivot col$^{\underline{m}}$s.

<u>Def$^{\underline{n}}$</u>: The no. of elements in a basis of a v.s. is called the <u>dimension</u> of the v.s.

This def$^{\underline{n}}$ has actually assumed the following theorem.

<u>Theorem</u>: If $v_1, v_2, \ldots, v_m$ and $w_1, w_2, \ldots, w_n$ are both bases for the same v.s. V, then $n = m$.

<u>Proof</u>: If possible let $n > m$.

We can write

$$\begin{aligned} w_1 &= a_{11} v_1 + \cdots + a_{m1} v_m \\ w_2 &= a_{12} v_1 + \cdots + a_{m2} v_m \\ &\vdots \\ w_n &= a_{1n} v_1 + \cdots + a_{mn} v_m \end{aligned}$$

This gives us the system

$$W = VA \quad \text{where } W = (w_1 \; w_2 \; \ldots \; w_n)$$

$$V = (v_1 \; v_2 \; \ldots \; v_m)$$

$$A = (a_{ij})_{\substack{i = 1, 2, \ldots, m \\ j = 1, 2, \ldots, n}}$$

A is of order/size $m \times n$ with $n > m$.

By our previous lecture we now know that $Ax = 0$ has a non-zero sol$^{\underline{n}}$.

$\Rightarrow VAx = 0 \Rightarrow Wx = 0$ ~ A combination of the basis elements giving us 0. (A contradiction)

If $m > n$ just reverse the role of V and W. //

Corollary: In a subspace of dimension k, no set of more than k vectors can be independent and/or can span the space.

Any linearly independent set in a v.s. V can be extended to a basis by adding more vectors if necessary.

A basis is a <u>maximal independent set</u>.

Any spanning set in V can be reduced to a basis, by discarding vectors if necessary.

A basis is a <u>minimal spanning set</u>.

Ex: Find a basis for $N(A)$ and $\dim N(A)$ when

$$A = \begin{pmatrix} -2 & 4 & -2 & -4 \\ 2 & -6 & -3 & 1 \\ -3 & 8 & 2 & -3 \end{pmatrix}.$$

Soln: By defn, $N(A)$ is the soln set of $Ax = 0$.

From A by row reduction we obtain,

$$R = \begin{pmatrix} 1 & 0 & 6 & 5 \\ 0 & 1 & 5/2 & 3/2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

pivot colns

The general sol^n for $Rx = 0$ i.e. $R\cdot\begin{pmatrix}u\\v\\w\\y\end{pmatrix} = 0$ is

then $\left.\begin{array}{l} u + 6w + 5y = 0 \\ v + \frac{5}{2}w + \frac{3}{2}y = 0 \end{array}\right\} \Rightarrow \begin{array}{l} u = -6w - 5y \\ v = -\frac{5}{2}w - \frac{3}{2}y \end{array}$

So, $x = \underbrace{\begin{pmatrix}-6\\-5/2\\1\\0\end{pmatrix}}_{v_1} w + \underbrace{\begin{pmatrix}-5\\-3/2\\0\\1\end{pmatrix}}_{v_2} y$

The vectors v_1 & v_2 span $N(A)$. They are linearly indep.

So, Basis $= \{v_1, v_2\}$, $\dim N(A) = 2$. //

In general the dimension of $N(A)$ is the no. of free parameters/variables in the sol^n set of $Ax = 0$.

i.e. $\dim N(A) = d = n - \text{rank}(A)$.

Ex: Let $A = \begin{pmatrix} 1 & 2 & 3 & -4 & 8 \\ 1 & 2 & 0 & 2 & 8 \\ 2 & 4 & -3 & 10 & 9 \\ 3 & 6 & 0 & 6 & 9 \end{pmatrix}$. Find a basis for $C(A)$ and $\dim C(A)$.

Sol^n: By def^n, the col^m space $C(A)$ is the span of the col^m of A. Let $A = (v_1\ v_2\ v_3\ v_4\ v_5)$ v_i's are col^{ms}.

We can now use trial and error to find the largest subset of col^n of A that are linearly independent.

eg. First we check if $\{v_1, v_2\}$ is independent.

- if yes then check if $\{v_1, v_2, v_3\}$ is indep.
- if not then discard v_2 and check if $\{v_1, v_3\}$.

Continue doing this until we exhaust all possibilities.

But there is a easier way. We find the row reduced echelon form R of A.

$$R = \begin{pmatrix} 1 & 2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

independent (by inspection).

$$c_2 = 2c_1, \quad c_4 = 2c_1 - 2c_3$$

These relations also hold for A.

$\Rightarrow C(A) = \text{span}\{v_1, v_3, v_5\}$ and $\dim C(A) = 3$ //.

This procedure also works in general.

The dim C(A) is the rank of A.

The dim N(A) is called the nullity of A.

The col^n space is also called the range space.

The row space of A is $C(A^T)$ ~ spanned by the rows of A.
$\dim C(A^T) = \text{rank}(A)$.

The <u>left nulspace</u> of A is $N(A^T)$. It contains all vectors y s.t. $A^T y = 0$.

The nullspace $N(A)$ and row space $C(A^T)$ are subspaces of $\mathbb{R}^n$.

The left null space $N(A^T)$ and col. space $C(A)$ are subspaces of $\mathbb{R}^m$.

(A is an $m \times n$ matrix; m col^n and n rows.)

The rank of a matrix also says something about the inverses.

Say we have an $m \times n$ matrix, A.

Then $\text{rank}(A) \leq m$, $\text{rank}(A) \leq n$.

If $\text{rank}(A) = m$ then a right-inverse of A exists.

If $\text{rank}(A) = n$ - - - left - - - - - -

Only a sq. matrix has both left & right inverses.

<u>Rank-Nullity Theorem:</u> Let A be an $m \times n$ matrix. The following eq^n holds: $n = \text{rank}(A) + \text{nullity}(A)$.

<u>Proof:</u> A basis for $C(A)$ can be computed by finding r.r.e.f. R. If r is the no. of leading ones in R then, $r = \text{rank}(A)$.

Let $d = n - r$, then the no. free parameters in the sol^n set of $Ax = 0$ is d and so a basis for $N(A)$

contains d vectors.

Thus, nullity $(A) = n - r = n - \text{rank}(A)$. //.

Ex: Find the rank & nullity of $A = \begin{pmatrix} 1 & -2 & 2 & 3 & -6 \\ 0 & -1 & -3 & 1 & 1 \\ -2 & 4 & -3 & -6 & 11 \end{pmatrix}$.

Sol^n: $A \xrightarrow{2R_1+R_3} \begin{pmatrix} 1 & -2 & 2 & 3 & 6 \\ 0 & -1 & -3 & 1 & 1 \\ 0 & 0 & 1 & 0 & -1 \end{pmatrix}$.

There are $r = 3$ leading entries, so $\text{rank}(A) = 3$.

nullity $(A) = n - \text{rank}(A) = 5 - 3 = 2$. //.

Theorem: Let A be an $n \times n$ matrix. The following statements are equivalent:

(i) the col^{ns} of A form a basis for $\mathbb{R}^n$

(ii) $C(A) = \mathbb{R}^n$

(iii) $\text{rank}(A) = n$

(iv) $N(A) = \{0\}$

(v) nullity $(A) = 0$

(vi) A is an invertible matrix.

Proof: Just review this lecture & the previous lecture.

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