Sequences and Series:

- We are mainly concerned with infinite series/sequences.

  eg. $\sum_{n=0}^{\infty} \frac{1}{2^n}$, $\sum_{n=0}^{\infty} \left(\frac{-1}{2}\right)^n$, etc.

Defn: A sequence is a fn. whose domain is $\mathbb{N}$.

eg: $(1, 1/2, 1/3, \ldots)$, $(F_1, F_2, F_3, \ldots)$ where $F_n = F_{n-1} + F_{n-2}$ $\forall\, n \geq 2$, etc.

Defn: A seqn (a_n) converges to a real no. ~~a~~ a, if for every positive no. ε, there exists an $N \in \mathbb{N}$ s.t. whenever $n \geq N$ we have $|a_n - a| < \varepsilon$.

We write this as $(a_n) \to a$ or $\lim_{n\to\infty} a_n = a$.

Defn: Given $a \in \mathbb{R}$, $\varepsilon > 0$, the set $V_\varepsilon(a) = \{x \in \mathbb{R} \mid |x - a| < \varepsilon\}$ is called the ε-neighbourhood of a.

Recasting now we see that, a seqn (a_n) converges to a if, given any ε-nbhd $V_\varepsilon(a)$ of a, there exists a pt in the seqn after which all terms are in $V_\varepsilon(a)$.

eg: $\lim a_n = 0$ where $a_n = 1/\sqrt{n}$.

Proof: Let $\varepsilon > 0$ be an arbitrary positive no. Choose $N \in \mathbb{N}$ s.t. $N > 1/\varepsilon^2$. Clearly, $n > 1/\varepsilon^2 \Rightarrow \frac{1}{\sqrt{n}} < \varepsilon$. for all $n \geq N$.

So, $|a_n - 0| < \varepsilon$. //

Proof Idea to show $x_n \to x$:

- Let $\varepsilon > 0$ be arbitrary,
- choose $N \in \mathbb{N}$
- Show that N chosen works.
- Assume $n \geq N$.
- Then show $|x_n - x| < \varepsilon$.

eg: $\lim \left(\frac{n}{n+1}\right) = 1$. (Choose $N > 1/\varepsilon$. $\Rightarrow n \geq N \Rightarrow n > 1/\varepsilon \Rightarrow 1/n < \varepsilon$.)

Thm: The limit of a seqn when it exists is unique.

Proof: Let us assume that a seq^n has more than one limit, if possible, say $a_n \to l_1$ and $a_n \to l_2$, with $l_1 \neq l_2$.

Let $\varepsilon = \frac{1}{3}|l_1 - l_2| > 0$.

Now, choose N_1 s.t. $|a_n - l_1| < \varepsilon \ \forall\, n \geq N_1$.

choose $N_2 \in \mathbb{N}$ s.t. $|a_n - l_2| < \varepsilon \ \forall\, n \geq N_2$.

Let $m_0 > \max(N_1, N_2)$, then $|a_{m_0} - l_1| < \varepsilon$ and

$|a_{m_0} - l_2| < \varepsilon$.

Now, $|l_1 - l_2| = |l_1 - a_{m_0} + a_{m_0} - l_2|$

$\leq |l_1 - a_{m_0}| + |a_{m_0} - l_2|$ (Triangle inequality)

$< \varepsilon + \varepsilon = 2\varepsilon = \frac{2}{3}|l_1 - l_2|$

$\Rightarrow |l_1 - l_2| < 0$ which is not possible. $\#$

Hence $l_1 = l_2$. //

- We say that a_n ~~are~~ converges to a if $a_n \to a$.
- If the seq^n (a_n) has no limit then we say that (a_n) is divergent. eg: $(1,2,1,2,\ldots)$, $(2,4,6,8,\ldots)$, $(-1,-4,-9,\ldots)$, etc.

Def^n: A seq^n (x_n) is bounded if there exists a no. $M > 0$ s.t. $|x_n| \leq M \ \forall\, n \in \mathbb{N}$.

Thm: Every convergent seq^n is bounded.

Proof: Let $(a_n) \to l$. Choose $\varepsilon = 1$ to find $N \in \mathbb{N}$ s.t. $|a_n - l| < 1$, for all $n \geq N$. i.e. $l - 1 < a_n < l + 1 \ \forall\, n \geq N$.

i.e. The set $\{a_N, a_{N+1}, a_{N+2}, \ldots\}$ is bounded.

Again, the set $\{a_1, a_2, \ldots, a_{N-1}\}$ is bounded by $\max\{a_i \mid i = 1, 2, \ldots, N-1\}$.

So, we get our result. //

Thm: Let $\lim a_n = a$ and $\lim b_n = b$. Then,

(i) $\lim (ca_n) = ca$, $\forall\ c \in \mathbb{R}$,

(ii) $\lim (a_n + b_n) = a+b$,

(iii) $\lim (a_n b_n) = ab$;

(iv) $\lim (a_n/b_n) = a/b$ provided $b \neq 0$.

Proof: (i) Let $c \neq 0$, let $\varepsilon > 0$ be arbitrary positive no.

Now, $|ca_n - ca| = |c|\,\underbrace{|a_n - a|}$.

we can make this as small as we want as $a_n \to a$.

So choose N s.t. $|a_n - a| < \varepsilon/|c|$ whenever $n \geq N$.

For all $n \geq N$, $|ca_n - ca| = |c|\,|a_n - a| < |c| \, \varepsilon/|c| = \varepsilon$.

For $c = 0$, we have $(0, 0, \ldots) \to 0$. //.

(ii) Note, $|(a_n + b_n) - (a+b)| \leq |a_n - a| + |b_n - b|$.

Since $a_n \to a$ and $b_n \to b$ we can choose N_1 and N_2 s.t.

$|a_n - a| < \varepsilon/2$ whenever $n \geq N_1$ and $|b_n - b| < \varepsilon/2$ whenever $n \geq N_2$.

Now, choose $N = \max\{N_1, N_2\}$. //.

(iii) Observe,
$$|a_n b_n - ab| = |a_n b_n - ab_n + ab_n - ab| \leq |a_n b_n - ab_n| + |ab_n - ab| = |b_n|\,|a_n - a| + |a|\,|b_n - b|.$$

Now choose N_1 s.t. $n \geq N_1 \Rightarrow |b_n - b| < \frac{\varepsilon}{|a| \cdot 2}$ for $a \neq 0$.

Since convergent seq^{n}s are bounded so, we know $\exists\ M > 0$ s.t. $|b_n| \leq M\ \forall\ n \in \mathbb{N}$. Choose N_2 so that $|a_n - a| < \frac{\varepsilon}{M \cdot 2}$ whenever $n \geq N_2$.

Pick $N = \max\{N_1, N_2\}$. //

(iv) This will follow from (iii) if we can prove that $b_n \to b \Rightarrow \frac{1}{b_n} \to \frac{1}{b}$ for $b \neq 0$.

observe, $\left|\frac{1}{b_n} - \frac{1}{b}\right| = \frac{|b-b_n|}{|b|\,|b_n|}$. Since $b_n \to b$ so we can make the numerator as small as we like by choosing $n \to \infty$.

Choose $\varepsilon_0 = |b|/2$. Since $b_n \to b$ so there exists some N_1 s.t. $|b_n - b| < \frac{|b|}{2}$ $\forall\, n \geq N_1$. This implies $|b_n| > \frac{|b|}{2}$.

Choose N_2 s.t. $n \geq N_2 \Rightarrow |b_n - b| < \frac{\varepsilon |b|^2}{2}$.

Let $N = \max\{N_1, N_2\}$, then $n \geq N \Rightarrow \left|\frac{1}{b_n} - \frac{1}{b}\right| = |b-b_n| \frac{1}{|b|\,|b_n|}$

$$< \frac{\varepsilon |b|^2}{2} \cdot \frac{1}{|b| \cdot \frac{|b|}{2}} = \varepsilon. \,//.$$
