Linear Transformations:

- We have been taking our scalars from ~~$\mathbb{Q}$~~ $\mathbb{R}$ so far.
- But we can easily take them from $\mathbb{Q}$ or $\mathbb{C}$.
- e.g. $\mathbb{Q}^n$ would be a v.s. with scalars from $\mathbb{Q}$.
  $\mathbb{C}^n$ . . . . $\mathbb{C}$.

The sets $\mathbb{R}, \mathbb{Q}, \mathbb{C}$ are examples of a <u>field</u>. This is a set together with an add^n and a mult^n map, such that the set contains elements that behave like zero and one and we can divide by any-non-zero element.

- In general we can take a v.s. over any field F. That is our set of scalars is now F. All of the concepts that we have learned so far remain valid for this more general notion of a v.s.

- Given any two v.s. V and W, we can also consider maps $T: V \to W$ from V to W that transform the add^n and ~~m~~ scalar mult.^{n} of V into that of W.

<u>Def^n:</u> Let V and W be v.s. over the same field F. We call a map $T: V \to W$ a <u>linear transformation (or, linear map) from V to W</u> if for all $x, y \in V$ and $a \in F$ we have:

(a) $T(x+y) = T(x) + T(y)$, and
(b) $T(ax) = aT(x)$.

<u>eg</u>: $T: \mathbb{R}^2 \to \mathbb{R}^2$ defined as $T(x) = \begin{pmatrix} x_1^2 \sin x_2 - \cos(x_1^2 - 1) \\ x_1^2 + x_2^2 + 1 \end{pmatrix}$ is not a linear map. <u>eg</u> $T\left(\begin{pmatrix}1\\0\end{pmatrix}\right) = \begin{pmatrix}-1\\2\end{pmatrix}$.

If T is linear then $T\left(\begin{pmatrix}3\\0\end{pmatrix}\right) = T\left(3\begin{pmatrix}1\\0\end{pmatrix}\right) = 3T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}-3\\6\end{pmatrix}$.

But, $T\left(\begin{pmatrix}3\\0\end{pmatrix}\right) = \begin{pmatrix}-\cos 8\\10\end{pmatrix} \neq \begin{pmatrix}-3\\6\end{pmatrix}$. //

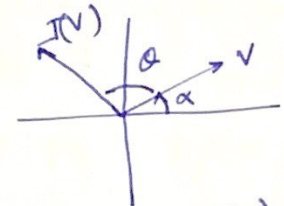
eg: $T: \mathbb{R}^2 \to \mathbb{R}^3$ defined by $T\left(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right) = \begin{pmatrix} 2x_1 - x_2 \\ x_1 + x_2 \\ -x_1 - 3x_2 \end{pmatrix}$ is linear.

e.g.: Let $\alpha \geq 0$, define $T: \mathbb{R}^n \to \mathbb{R}^n$ by $T(x) = \alpha x$. If $0 \leq \alpha \leq 1$, then T is called a contraction and if $\alpha > 1$ then T is called a dilation.

Ex: Let $T: \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation s.t. $T(u) = (3, 4)$ and $T(v) = (-2, 5)$. What is $T(2u + 3v)$?

Ex: Let $T_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be a map on the 2D-plane that rotates every $v \in \mathbb{R}^2$ by an angle θ. Then T_θ is a linear map.

- If $v = (\cos\alpha, \sin\alpha)$ then



$T_\theta(v) = \left(\cos(\alpha + \theta), \sin(\alpha + \theta)\right)$.

$= \left(\cos\alpha\cos\theta - \sin\alpha\sin\theta,\ \cos\alpha\sin\theta + \sin\alpha\cos\theta\right)$

$= \cancel{\cos\theta \ \sin\theta} \underbrace{\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}}_{T} \underbrace{\begin{pmatrix} \cos\alpha \\ \sin\alpha \end{pmatrix}}_{v}$.

$T_\theta(cv) = c\, T_\theta(v)$

$T_\theta(u + v) = T_\theta(u) + T_\theta(v)$. //.

Proposition: Let V, W be v.s. over the same field F and let $T: V \to W$ be a map. Then T is a linear map iff $T(ax + y) = aT(x) + T(y)$ for all $x, y \in V$ and $a \in F$.

Proof: Just use the def^{n}.

Def^{n}: The null space of T, denoted by $N(T)$ is defined as

$$N(T) = \{ v \in V \mid T(v) = 0 \}.$$

The range space of T is denoted by $\mathrm{Im}(T)$ and defined as,

$$\mathrm{Im}(T) = \{ T(v) \in W \mid v \in V \}.$$

- A linear map is injective iff $N(T)=\{0\}$. $T: V \to W$
- A linear map is surjective iff $Im(T)=W$.
- $\text{nullity}(T) = \dim N(T)$, $\text{rank}(T) = \dim Im(T)$

Rank-Nullity Thm: $\text{nullity}(T) + \text{rank}(T) = \dim(V)$.

Thm: Let V and W be v.s. and $T: V \to W$ be a linear map. If $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for V, then

$$Im(T) = \text{span}(T(\beta)) = \text{span}\{T(v_1), \dots, T(v_n)\}.$$

Proof: Since $T(v_i) \in Im(T)$ so, $\text{span}(T(\beta)) \subset Im(T)$.

Let $w \in Im(T)$, then $\exists\, v \in V$ s.t. $w = T(v)$. As β is a basis for V so, $\exists\, a_1, a_2, \dots, a_n \in F$ s.t. $v = a_1 v_1 + \dots + a_n v_n$.

$$\therefore w = T(v) = a_1 T(v_1) + \dots + a_n T(v_n) \in \text{span}(T(\beta)).$$

Since w was arbitrary so, $Im(T) \subset \text{span}(T(\beta))$ //.

- A very important property of linear maps is that its action on an arbitrary vector $v \in V$ is completely determined by its action on some chosen basis of V.

Thm: Let V be a f.d.v.s. and let W be a v.s. over the field F. Let $\beta = \{v_1, v_2, \dots, v_n\}$ be a basis for V. Let $w_1, \dots, w_n \in W$ be arbitrary vectors of W. Then there exists a unique linear map $T: V \to W$ s.t. $T(v_i) = w_i$ for $i \in \{1, 2, \dots, n\}$.

Proof: For $x \in V$, $\exists\, a_1, \dots, a_n \in F$ s.t. $x = a_1 v_1 + \dots + a_n v_n$.

Define $T: V \to W$ by, $T(x) = a_1 w_1 + \dots + a_n w_n$.

If $x = v_i$ then $a_j = 0$ for all $j \neq i$ and $a_i = 1$. So, $T(v_i) = w_i$.

Clearly T is linear.

Let U be another linear map s.t. $U(v_i) = w_i$ $\forall\, i \in \{1, \dots, n\}$.

Then, $U(x) = \sum_{i=1}^{n} a_i U(v_i) = \sum_{i=1}^{n} a_i w_i = T(x)$. $\Rightarrow T = U$. //.

<u>Thm:</u> Let V and W be f.d. v.s. s.t. $\dim V = \dim W$. Let $T : V \to W$ be a linear map. Then the following are equivalent:

(a) T is injective,

(b) T is surjective,

(c) $\dim(\mathrm{Im}(T)) = \dim(V)$. //
