Indian Institute of Information Technology (IIIT) Manipur
End Semester Examination, February 2023

Course Title: **Mathematics I** Course Code: **MA1011/MA101**
Semester: I Maximum Marks: 100
Date of Examination: 24 February 2023 Time: 3 hours

The number in the brackets indicate the marks to be awarded for completing that particular step in the solution.
The solutions are indicative only, we will accept other valid solutions as well.

Part A (10×2 marks $= 20$ marks)

1. Given that $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix} A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, find A^{-1}.

   Solution. Let $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$ and $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$. We note that $P^{-1} = P^T$. **(1)**

   Multiply each side by P^{-1} to get the following $P^{-1}BA = I$. So $A^{-1} = P^{-1}B = \begin{pmatrix} 1 & 0 & 8 \\ 1 & 2 & 3 \\ 2 & 5 & 3 \end{pmatrix}$. **(1)**

2. Given that $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix} A = \begin{pmatrix} 23 & 0 & 0 \\ 0 & 0 & 23 \\ 0 & 23 & 0 \end{pmatrix}$, find the value of $\det(A)$.

   Solution. Let $B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 5 & 3 \\ 1 & 0 & 8 \end{pmatrix}$ and $P = \begin{pmatrix} 23 & 0 & 0 \\ 0 & 0 & 23 \\ 0 & 23 & 0 \end{pmatrix}$, then $\det(B) \cdot \det(A) = \det(P)$. **(1)**

   After calculating we get $\det(B) = -1$ and $\det(P) = -23^3$, so $\det(A) = 23^3$. **(1)**

3. Is there a matrix A for which $(1 \quad 2 \quad 1)^T$ is a basis for the column space and $(1 \quad 1 \quad 1)^T$ is a basis for the nullspace? If yes, what is A? if no, why does no such A exist?

   Solution. No. **(1)**

   Let $m = n = 3$, then A is an $m \times n$ matrix. Let the dimension of the column space be r, then the dimension of the nullspace is $n - r$. Here we are given $r = 1$ and $n - r = 1$ which is not correct by the rank-nullity theorem. **(1)**

4. Find the supremum and infimum of the following set

$$\left\{ 2(-1)^{n+1} + (-1)^{\frac{n(n+1)}{2}} \left(2 + \frac{3}{n} \right) : n \in \mathbb{N} \right\}.$$

Solution. The set can be written as

$$\left\{-2, -\frac{11}{2}, 5\right\} \cup \left\{\frac{3}{4k}, -\frac{3}{4k+1}, -4 - \frac{3}{4k+2}, 4 + \frac{3}{4k+2} : k \in \mathbb{N}\right\}.$$

The supremum is 5. (**1**)

The infimum is $-\frac{11}{2}$. (**1**)

5. Is the series $\sum_{i=1}^{\infty} \sin i$ convergent? Justify your answer.

   Solution. No. (**1**)

   $\sin(n) > 0.1$ for infinitely many n and $\sin(n) < -0.1$ for infinitely many n, so the terms do not go to any limit. (**1**)

6. Write the statement of Taylor's Theorem.

   Solution. Let $f : (a,b) \to \mathbb{R}$ be a $n+1$ times differentiable function and $x_0, x \in (a,b)$. Then there exists a point ξ, such that $x < \xi < x_0$, such that

   $$f(x) = P_n(x; x_0) + \frac{(x-x_0)^{n+1}}{(n+1)!} f^{(n+1)}(\xi),$$

   where $P_n(x; x_0)$ is the Taylor polynomial of order n about the point x_0 of the function f. (**2**)

   (**No part marks for this question.**)

7. If a function f has the property that for all real numbers x, we have $3 - |x| \leq f(x) \leq 3 + |x|$, then from this conclude $f(x) \to$ ______ as $x \to$ ______ .

   Solution. $f(x) \to 3$ as $x \to 0$ by Squeeze theorem. (**2**)

   (**No part marks for this question.**)

8. Evaluate $\int_0^3 f(x)\mathrm{d}x$, where $f(x) = \begin{cases} x^2, & x < 2 \\ 3x - 2, & x \geq 2. \end{cases}$

   Solution. We have $\int_0^3 f(x)\mathrm{d}x = \int_0^2 x^2 \mathrm{d}x + \int_2^3 (3x-2)\mathrm{d}x$. (**1**)

   Evaluating this we obtain $\left[\frac{x^3}{3}\right]_0^2 + \left[\frac{3x^2}{2} - 2x\right]_2^3 = \frac{49}{6}$. (**1**)

9. Find an interval $[a,b]$ on which $f(x) = x^4 + x^3 - x^2 + x - 2$ satisfies the hypothesis of the Rolle's theorem.

   Solution. Since the function is a polynomial so it is continuous and differentiable on the appropriate sets. We need to check for which values of a and b we have $f(a) = f(b) = 0$. (**1**)

   Solving $x^4 + x^3 - x^2 + x - 2 = 0$ we obtain as real roots $x = 1$ and $x = -2$. So $a = -2$ and $b = 1$. (**1**)

10. State the Cauchy convergence criterion and the monotone convergence theorem.

    Solution. Cauchy convergence criterion states that, a sequence is convergent if and only if it is a Cauchy sequence. (**1**)

    Monotone convergence theorem states that, if a sequence is monotone and bounded then it converges. (**1**)