Indian Institute of Information Technology (IIIT) Manipur
End Semester Examination, February 2023

Course Title: **Mathematics I** | Course Code: **MA1011/MA101**

Semester: I | Maximum Marks: 100

Date of Examination: 24 February 2023 | Time: 3 hours

- **The number in the brackets indicate the marks to be awarded for completing that particular step in the solution.**
- **The solutions are indicative only, we will accept other valid solutions as well.**
- **All steps are not shown here but it is expected that the student will carry out all steps for full credit.**

Part B (5×16 marks $= 80$ marks)

11. If $f(x) = x^{2/3}$, $a = -1$ and $b = 8$,

(I) Show that there is no point c in (a, b) such that $f'(c) = \dfrac{f(b) - f(a)}{b - a}$.

Solution. We have $f'(x) = \dfrac{2}{3x^{1/3}}$. **(2)**

And, $\dfrac{f(b) - f(a)}{b - a} = \dfrac{1}{3}$. **(2)**

If there exists a $c \in (-1, 8)$ such that $f'(c) = \dfrac{2}{3c^{1/3}} = \dfrac{1}{3}$ then c must necessarily be equal to 8 which is not in the interval $(-1, 8)$. Hence no such c exists. **(4)**

(II) Explain why the result in part (I) does not contradict the Mean-Value Theorem.

Solution. The prerequisites of the MVT is that the function f is continuous on the closed interval $[a, b]$, and differentiable on the open interval (a, b). **(2)**

Here the function is not differentiable at $x = 0$. **(2)**

We can prove this fact by using the limit definition of derivatives. (**Proof omitted here, but proof is required for full credit in the exam.**) **(4)**

12. (A) Given $A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ -1 & -1 \end{pmatrix}$.

(I) Apply the Gram-Schmidt process to the columns of the matrix A in the order that they occur in the matrix. Use this to write $A = QU$, where Q is a matrix with orthonormal columns and R is an upper triangular matrix. **(4+4 marks)**

Solution. We set $q_1 = a_1 = (1, 0, 0, -1)$. **(1)**

$q_2 = a_2 - \dfrac{(q_2 \cdot q_1)}{(q_1 \cdot q_1)} q_1 = (-1/2, 1, 0, -1/2)$. **(3)**

We normalize $q_1 = (1/\sqrt{2}, 0, 0, -1/\sqrt{2})$ and $q_2 = (-1/\sqrt{6}, 2/\sqrt{6}.0, -1/\sqrt{6})$. **(1)**

Set $Q = [q_1, q_2]$. **(1)**

Set $R = Q^T A$ to get $A = QR = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{6} \\ 0 & 2/\sqrt{6} \\ 0 & 0 \\ -1\sqrt{2} & -1\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{2} & 1/\sqrt{2} \\ 0 & 3/\sqrt{6} \end{pmatrix}$. **(2)**

(II) Compute the matrix of the projection onto the column space of A. What is the distance of the point $(1, 1, 1, 0)$ to this column space?

Solution. The required matrix is $P = QQ^T$. **(2)**

After calculating we obtain $P = \begin{pmatrix} 2/3 & -1/3 & 0 & -1/3 \\ -1/3 & 2/3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1/3 & -1/3 & 0 & 2/3 \end{pmatrix}$. **(2)**

If $b = (1, 1, 1, 0)$ then its projection is $p = Pb = (1/3, 1/3, 0, -2/3)$. **(2)**

The required distance is then the length of $b - p$, which is $\sqrt{21}/3$. **(2)**

OR

(B) Given $A = \begin{pmatrix} 1 & -6 & -4 \\ 0 & 4 & 2 \\ 0 & -6 & -3 \end{pmatrix}$.

(I) Find the eigenvalues of the matrix A.

Solution. To find the eigenvalue we need to solve the equation $\det(A - \lambda I) = 0$ for λ. **(1)**

The equation we obtain is $-\lambda^3 + 2\lambda^2 - \lambda = 0$. **(2)**

Solving this we obtain the following eigenvalues: $1, 1, 0$. **(1+1+1)**

(II) Diagonalize the matrix A. **(10 marks)**

Solution. We obtain the following $A = PBP^{-1}$, where

$$P = \begin{pmatrix} 2 & 0 & 1 \\ -1 & -2 & 0 \\ 2 & 3 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(**Getting these matrices would fetch 5 marks and the derivation of the process would fetch another 5 marks.**)

13. (A) Given that $f : (a, b) \to \mathbb{R}$ is a differentiable function.

(I) Prove that $f(x)$ is increasing on (a, b) iff $f'(x) \geq 0$ for all $x \in (a, b)$.

Solution. If f is monotone increasing then by definition $f(x) \leq f(y)$ if $x \leq y$ while $f(x) \geq f(y)$ if $x \geq y$. Then for increasing functions the Newton-Quotient is always non-negative so the derivative is always non-negative as it is the limit of a non-negative function. **(2)**

Conversely, since $f(x)$ is differentiable we can apply the MVT: given two points $x_1 < x_2$ in (a, b) we look at (x_1, x_2). By the MVT we have some point $x_0 \in (x_1, x_2)$ such that **(2)**

$$f'(x_0) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Since the derivative is non-negative we get

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_0) \geq 0,$$

which implies $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$. **(2)**

(II) Prove that if $f'(x) > 0$ for all $x \in (a, b)$, then $f(x)$ is strictly increasing.

Solution. We assume $f'(x) > 0$, that is

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} > 0.$$

Then, there exists an interval $(x_0 - \delta, x_0 + \delta)$ for some $\delta > 0$ such that $\frac{f(x) - f(x_0}{x - x_0} > 0$. This implies that $f(x)$ is monotone increasing in the interval $(x_0 - \delta, x_0 + \delta)$. **(4)**

Since this is true at any point $x_0 \in (a, b)$, we take the union of all the intervals $(x_0 - \delta, x_0 + \delta)$ to conclude that $f(x)$ is monotone strictly increasing in the whole domain. **(2)**

(III) Is the reverse implication of part (II) true? If yes, why? If no, why not?

Solution. No. **(1)**

Consider the example $f(x) = x^3$. **(1)**

This function is strictly increasing and differentiable on $\mathbb{R}$ but $f'(0) = 0$. **(2)**

OR

(B) (I) Show that the series $\sum_{n=0}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \leq 1$.

Solution. Let us denote the partial sums by

$$A_{n,p} := 1 + \frac{1}{2^p} + \cdots + \frac{1}{n^p}.$$

The series converges/diverges if the sequence of partial sums converges/diverges. **(1)**

Clearly for $p = 1$, $(A_{n,1})$ is a monotonically increasing sequence. **(1)**

Also for $n \in \mathbb{N}$ we have

$$\begin{aligned}
A_{2^n} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \cdots + \left(\frac{1}{2^{n-1} + 1} + \cdots + \frac{1}{2^n}\right) \\
&\geq 1 + \frac{1}{2} + \frac{2}{4} + \cdots + \frac{2^{n-1}}{2^n} \\
&= 1 + \frac{n}{2}.
\end{aligned}$$

This shows that $(A_{n,1})$ is not bounded above and hence is not convergent. **(1)**

For $p = 2$ we have

$$\begin{aligned}
A_{n,2} &\leq 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots + \frac{1}{(n-1) \cdot n} \\
&= 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\
&= 2 - \frac{1}{n} < 2.
\end{aligned}$$

Thus $(A_{n,2})$ is bounded above and is convergent. **(1)**
For any p we have that $(A_{n,p})$ is monotonically increasing. Since for $p = 1$ the sequence is divergent, so the given series is divergent for $p \leq 1$ since **(2)**

$$0 < \frac{1}{n} \leq \frac{1}{n^p} \quad \text{if} \quad p \leq 1.$$

Again since the sequence is convergent for $p = 2$ then given series is convergent for $p \geq 2$ since we have **(2)**

$$0 < \frac{1}{n^p} \leq \frac{1}{n^2} \quad \text{if} \quad p \geq 2.$$

(II) Show that the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$ is convergent for $-1 < x \leq 1$.
Solution. We apply the ratio test here. **(1)**
Define $L := \lim_{n\to\infty} \left| \frac{(-1)^{n+2} x^{n+1}/(n+1)}{(-1)^{n+1} x^n / n} \right|$. **(1)**
Calculate L. **(2)**
Conclude based on L that the series converges for $|x| < 1$, $x \neq -1$. **(3)**
Justify why the series does not converge when $x = -1$. **(1)**

14. (A) (I) Given that f is continuous on $[a, b]$, $a > 0$ and differentiable on (a, b). Show that if

$$\frac{f(a)}{a} = \frac{f(b)}{b},$$

then there exists an $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$.
Solution. We need to apply Rolle's Theorem to the function $h(x) = \frac{f(x)}{x}$, $x \in [a, b]$.
(**Identifying the correct function fetches 2 marks and the rest of the derivation fetches 4 marks.**)

(II) Given that f is continuous on $[0, 2]$ and twice differentiable on $(0, 2)$. Show that if

$$f(0) = 0, \quad f(1) = 1 \quad \text{and} \quad f(2) = 2,$$

then there exists an $x_0 \in (0, 2)$ such that $f''(x_0) = 0$.
Solution. From the MVT, there exists some $x_1 \in (0, 1)$ and $x_2 \in (1, 2)$ such that **(3)**

$$f'(x_1) = f(1) - f(0) = 1 \quad \text{and} \quad f'(x_2) = f(2) - f(1) = 1.$$

Now we apply Rolle's Theorem to f' on $[x_1, x_2]$. **(3)**

(III) Given that $f : \mathbb{R} \to \mathbb{R}$ is $n + 1$ times differentiable on $\mathbb{R}$. Prove that for every $x \in \mathbb{R}$ there is a $\theta \in (0, 1)$ such that

$$f(x) = f(0) + xf'(x) - \frac{x^2}{2} f''(x) + \cdots + (-1)^{n+1} \frac{x^n}{n!} f^{(n)}(x) + (-1)^{n+2} \frac{x^{n+1}}{(n+1)!} f^{(n+1)}(\theta x),$$

and

$$f\left(\frac{x}{1+x}\right) = f(x) - \frac{x^2}{1+x}f'(x) + \cdots + (-1)^n \frac{x^{2n}}{(1+x)^n}\frac{f^{(n)}(x)}{n!}$$
$$+ (-1)^{n+1}\frac{x^{2n+2}}{(1+x)^{n+1}}\frac{f^{(n+1)}\left(\frac{x+\theta x^2}{1+x}\right)}{(n+1)!}, \quad x \neq -1.$$

Solution. By Taylor's theorem we have **(1)**

$$f(0) = f(x + (-x)) = f(x) + \frac{f'(x)}{1!}(-x) + \cdots + \frac{f^{(n)}(x)}{n!}(-x)^n + \frac{f^{(n+1)}(x - x_0)}{(n+1)!}(-x)^{n+1}.$$

Now take x_0 such that $\theta = 1 - \dfrac{x_0}{x}$. **(1)**

For the next part, observe $f\left(\dfrac{x}{1+x}\right) = f\left(x - \dfrac{x^2}{1+x}\right)$. **(1)**

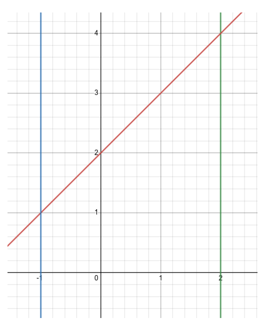
Now proceed like before. **(1)**

OR

(B) (I) Sketch the region whose area is represented by the following definite integrals, and evaluate the integral using an appropriate formula:

(a) $\int_{-1}^{2} (x+2)\mathrm{d}x$,

Solution. The region is the one bounded by the lines in red, blue, green and the x-axis as shown below: **(2.5)**

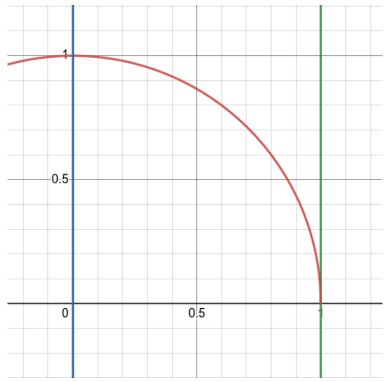


We have **(2.5)**

$$\int_{-1}^{2} (x+2)\mathrm{d}x = \left[\frac{x^2}{2} + 2x\right]_{-1}^{2} = \frac{15}{2}.$$

(b) $\int_{0}^{1} \sqrt{1-x^2}\mathrm{d}x$.

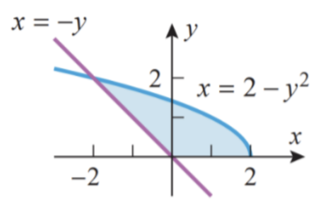
Solution. The region is the one bounded by the curve in red, the lines in blue, green and the x-axis as shown below: **(2.5)**



We have **(2.5)**

$$\int_0^1 \sqrt{1-x^2}\mathrm{d}x = \frac{1}{2}\left[x\sqrt{1-x^2}+\arcsin(x)\right]_0^1 = \frac{\pi}{4}.$$

(II) Find the area of the shaded region in the figure given below.



Solution. The required area is **(2)**

$$\int_0^2 \{(2-y^2)+y\}\mathrm{d}y.$$

The value of the integral is **(2)**

$$\left[2y - \frac{y^3}{3} + \frac{y^2}{2}\right]_0^2$$

So, the required area is $\dfrac{10}{3}$. **(2)**

15. (A) (I) Find an antiderivative of $f:\mathbb{R}\to\mathbb{R}$ defined by $f(\theta)=\sin^4\theta\cos^2\theta$. **(5 marks)**
Solution. We have for any $a\in\mathbb{R}$, **(1)**

$$\int_a^x \sin^4\theta\cos^2\theta\mathrm{d}\theta = \frac{1}{2}\int_a^x (1-\cos(2\theta)\left(\frac{1}{2}\sin(2\theta)\right)^2\mathrm{d}\theta.$$

Simplifying further we eventually obtain **(1)**

$$\int_a^x \sin^4\theta\cos^2\theta \mathrm{d}\theta = \frac{1}{8}\int_a^x \frac{1}{2}(1-\cos(4\theta)\mathrm{d}\theta - \int_a^x \sin^2(2\theta)\cos(2\theta)\mathrm{d}\theta.$$

For the second integral we use the substitution $u = \sin(2\theta)$ so that $u'(\theta) = 2\cos(2\theta)$ and $\cos(2\theta)\mathrm{d}\theta = \frac{1}{2}\mathrm{d}u$. **(1)**

Using this substitution we will obtain **(1)**

$$\begin{aligned}\int_a^x \sin^4\theta\cos^2\theta \mathrm{d}\theta &= \frac{1}{8}\left[\frac{1}{2}\theta - \frac{1}{8}\sin(4\theta)\right]_a^x - \frac{1}{16}\left[\frac{u^3}{3}\right]_{\sin(2a)}^{\sin(2x)} \\ &= \frac{1}{16}x - \frac{1}{64}\sin(4x) - \frac{1}{16}a + \frac{1}{64}\sin(4a) - \frac{1}{48}\sin^3(2x) + \frac{1}{48}\sin^3(2a),\end{aligned}$$

and an antiderivative $F:\mathbb{R}\to\mathbb{R}$ is given by **(1)**

$$F(\theta) = \frac{1}{16}\theta - \frac{1}{64}\sin(4\theta) - \frac{1}{48}\sin^3(2\theta).$$

(**Note that the final expression can be written in many different ways using double-angle formulas, we will accept all valid solutions.**)

(II) Evaluate the integral $\int_0^{\pi/2} \frac{1+\cos\theta}{3-\cos\theta}\mathrm{d}\theta$.

Solution. (**Each line of simplification is worth 1 mark in this solution.**)

$$\begin{aligned}\int_0^{\pi/2} \frac{1+\cos\theta}{3-\cos\theta}\mathrm{d}\theta &= \int_{u(0}^{u(\pi/2)} \frac{1+\frac{1-u^2}{1+u^2}}{3-\frac{1-u^2}{1+u^2}}\cdot\frac{2}{1+u^2}\mathrm{d}u \\ &= 2\int_{\tan 0}^{\tan(\pi/4)} \frac{1}{1+2u^2}\cdot\frac{1}{1+u^2}\mathrm{d}u \\ &= 4\int_0^1 \frac{1}{1+2u^2}\mathrm{d}u - 2\int_0^1\frac{1}{1+u^2}\mathrm{d}u \\ &= \frac{4}{\sqrt{2}}[\arctan(\sqrt{2}u)]_0^1 - 2[\arctan u]_0^1 \\ &= \frac{4}{\sqrt{t2}}\arctan\sqrt{2} - \frac{\pi}{2}.\end{aligned}$$

(III) Find the values of $\alpha\in\mathbb{R}$ for which the integral $\int_1^\infty x^\alpha \mathrm{d}x$ converges.

Solution. Let $\alpha \neq -1$, then **(1)**

$$\int_1^\infty x^\alpha\mathrm{d}x = \lim_{b\to\infty}\int_1^b x^\alpha\mathrm{d}x = \lim_{b\to\infty}\left[\frac{x^{\alpha+1}}{\alpha+1}\right]_1^b = \frac{1}{\alpha+1}\lim_{b\to\infty}(b^{\alpha+1}-1).$$

Now $\lim_{b\to\infty} b^{\alpha+1} = +\infty$ for $\alpha+1>0$, so $\lim_{b\to\infty}(b^{\alpha+1}-1) = +\infty$. **(1)**

For $\alpha < -1$ we have negative power of b hence $\lim_{b\to\infty} b^{\alpha+1} = 0$. **(1)**

For $\alpha = -1$ we have **(1)**

$$\int_1^\infty \frac{1}{x}\mathrm{d}x = \lim_{b\to 1}\int_1^x \frac{1}{x}\mathrm{d}x = \lim_{b\to\infty} \ln b = \infty.$$

So we get that the integral converges if and only if $\alpha < -1$ and the value of the integral is then **(2)**

$$\int_1^\infty x^\alpha \mathrm{d}x = \frac{1}{\alpha+1}\lim_{b\to\infty}(b^{\alpha+1}-1) = \frac{1}{\alpha+1}(-1) = -\frac{1}{\alpha+1}.$$

OR

(B) (I) Find the volume of the solid generated when the region enclosed by $y = \sqrt{x}$, $y = 2$, and $x = 0$ is revolved about the y-axis.

Solution. The required volume is given by the integral $\int_0^2 \pi[u(y)]^2\mathrm{d}y = \int_0^2 \pi y^4 \mathrm{d}y = \frac{32\pi}{5}$ sq. units. (**Identifying the integral correctly will fetch 4 marks. And, then correctly evaluating this integral would fetch another 4 marks.**)

(II) Find the exact arc length of the curve $y = 3x^{3/2} - 1$ from $x = 0$ to $x = 1$.

Solution. The exact arc-length of $f(x)$ between a and b is given by $\int\limits_a^b \sqrt{1 + f'(x)^2}\mathrm{d}x$. **(1)**

We have $y = 3x^{3/2} - 1$, so $y' = \frac{9}{2}x^{1/2}$. **(1)**

Substituting this with $a = 0$ and $b = 1$ in the formula, we obtain **(2)**

$$\int_0^1 \sqrt{1 + \left(\frac{9}{2}x^{1/2}\right)^2}\mathrm{d}x = \frac{1}{2}\int_0^1 \sqrt{81x+4}\mathrm{d}x.$$

Now we substitute $81x + 4 = u^2$ to get $\mathrm{d}x = \frac{2}{81}u\mathrm{d}u$ and the integral becomes **(2)**

$$\frac{1}{2}\int_0^1 \sqrt{u^2}\cdot\frac{2}{81}u\mathrm{d}u.$$

Evaluating this integral we obtain the arc length to be $\frac{1}{243}$. **(2)**