MA1011: Problem Sheet 3 (Linear Independence, Basis & Dimension)

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Date of Submission

22 December 2022 by 1200 IST. If I am not in the office (F-7) then please slide your submission under the door.

General Rules

- This problem sheet will be graded, the numbers in the brackets denote the points for each question.
- You can work in groups and you are free to consult any material that you wish to, but please mention them when you write down your answers/solutions. You must also mention your roll number and branch at the top of your submission.
- Write legibly and in full sentences. Mathematics, like any other subject has its own style and diction. Our aim is also to learn how to write mathematics (and other technical materials).

Problems

1. Let $v_1, v_2, \dots, v_k$ be linearly dependent vectors in the vector space V. Then prove that there exists some $j \in \{1, 2, \dots, k\}$ such that
   (a) $v_j \in \operatorname{span}\{v_1, v_2, \dots, v_{j-1}\}$; and
   (b) if the jth term is removed from $v_1, v_2, \dots, v_k$ then the span of the remaining terms equals $\operatorname{span}\{v_1, v_2, \dots, v_k\}$.

   [**2+2 points**]
2. Prove that every subspace of a finite-dimensional vector space is finite-dimensional. [**3 points**]
3. Prove that every linearly independent list of vectors in a finite-dimensional vector space can be extended to a basis of the vector space. [**3 points**]
4. Let V be the plane in $\mathbb{R}^3$ spanned by the vectors $v_1 = (1, -2, 1)^T$ and $v_2 = (2, -3, 1)^T$. Is the vector $v = (0, 1, -1)^T$ an element of V? [**1 point**]
5. Find a basis of the vector space $\mathbb{P}_n[t]$ described in Problem Sheet 1. [**2 points**]
6. Is the following set a basis for $\mathbb{R}^4$: $\{(1, 1, 1, 1)^T, (1, 1, -1, -1)^T, (1, -1, 0, 0)^T, (0, 0, 1, -1)^T\}$? [**2 points**]
7. A 3×3 matrix is said to be a semi-magic square if its row sums and column sums (i.e., the sum of entries in an individual row or column) all add up to the same number. Explain why the set of all semi-magic squares is a subspace of the vector space of 3×3 matrices. Prove that the 3×3 permutation matrices span the space of semi-magic squares. What is its dimension? [**2+2+1 points**]
8. Let $v_1, v_2, \dots, v_n$ be a basis for $\mathbb{R}^n$. Prove that, if A is a non-singular matrix then $Av_1, Av_2, \dots, Av_n$ is also a basis for $\mathbb{R}^n$. What is this basis when $v_i = e_i$? [**3+1 points**]
9. Find the nullspace and its dimention for the matrix

$$\begin{pmatrix} 1 & -2 & 0 & 3 \\ 2 & -3 & -1 & -4 \\ 3 & -5 & -1 & -1 \end{pmatrix}.$$

   [**3 points**]