MA1011: Problem Sheet 5 (Determinants)

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December 2022

Date of Submission

28 December 2022 by 1200 IST. If I am not in the office (F-7) then please slide your submission under the door.

General Rules

- This problem sheet will be graded, the numbers in the brackets denote the points for each question.
- You can work in groups and you are free to consult any material that you wish to, but please mention them when you write down your answers/solutions. You must also mention your roll number and branch at the top of your submission.

Problems

1. Write an algorithm to determine the determinant of a matrix using the approach discussed at the end of the determinants section (the one involving permutation matrices). [**6 points**]
2. We will find the determinant of the matrix $D = (x_j^{n-i})_{1\leq i,j\leq n}$ using the following steps:
   (a) Write down the matrix D for $n = 3$ and calculate its determinant. [**2+2 points**]
   (b) Functions that change sign when we transpose any two of the variables are called *alternating functions*. Show that $\det D$ is an alternating function in the variables $x_1, x_2, \ldots, x_n$. [**3 points**]
   (c) Show that the degree of the alternating function described above is $n(n-1)/2$. [**4 points**]
   (d) Show that $\prod_{1\leq i<j\leq n}(x_i - x_j)$ is an alternating polynomial of degree $n(n-1)/2$. [**3 points**]
   (e) Show that $\det D = C \times \prod_{1\leq i<j\leq n}(x_i - x_j)$ for some constant C by using the following result: if $f(x_1, x_2, \ldots, x_n)$ is an alternating function of degree d, then $\dfrac{f(x_1, x_2, \ldots, x_n)}{\prod_{1\leq i<j\leq n}(x_i - x_j)}$ is a symmetric polynomial[1] of degree $d - n(n-1)/2$. [**6 points**]
   (f) Compare the coefficient of $x_1^{n-1}x_2^{n-2}\cdots x_{n-1}$ on both sides of the equation
   $$\det D = C \times \prod_{1\leq i<j\leq n}(x_i - x_j)$$
   to show that $C = 1$. [**3 points**]
3. Using the process described above show that the determinant of the matrix
   $$K = ((x_1 + a_n)(x_i + a_{n-1})\cdots(x_i + a_{j+1})(x_i + b_j)(x_i + b_{j-1})\cdots(x_i + b_2))_{1\leq i,j\leq n}$$
   is equal to $\prod_{1\leq i<j\leq n}(x_i - x_j)\prod_{2\leq i\leq j\leq n}(b_i - a_j)$. [**10 points**]

[1] A polynomial is symmetric if any of the variables are interchanged, we obtain the same polynomial.