MA1011: Problem Sheet 6 (Inner Products & Orthogonality)

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Date of Submission

09 January 2023 by 1200 IST. If I am not in the office (F-7) then please slide your submission under the door. Make sure to staple any loose sheets of paper.

General Rules

- This problem sheet will be graded, the numbers in the brackets denote the points for each question.
- You can work in groups and you are free to consult any material that you wish to, but please mention them when you write down your answers/solutions. You must also mention your roll number, section and branch at the top of your submission.

Problems

1. Explain why the formula for the length of a vector in $\mathbb{R}^2$ follows from the Pythagorean Theorem. How do you use the Pythagorean Theorem to justify the formula for the length of a vector in $\mathbb{R}^3$? [**2+2 points**]
2. Find the value(s) of a for which the vector $(2, a, -3)^T$ is orthogonal to $(-1, 3, -2)^T$. Is there any value of a for which $(2, a, -3)^T$ is parallel to $(-1, 3, -2)^T$? [**2+2 points**]
3. Let $v = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ and $w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ be two vectors in $\mathbb{R}^3$. The cross product between two vectors in $\mathbb{R}^3$ is defined as the vector
$$v \times w = \begin{pmatrix} v_2w_3 - v_3w_2 \\ v_3w_1 - v_1w_3 \\ v_1w_2 - v_2w_1 \end{pmatrix}.$$
Prove that $v \times w$ is orthogonal to both v and w. Further, if v and w are orthogonal vectors then show that $v \times w, v, w$ form an orthogonal basis of $\mathbb{R}^3$. [**2+2 points**]
4. Let $v_1, v_2, \ldots, v_n$ be an orthogonal basis of a vector space V and let $v = a_1v_1 + a_2v_2 + \cdots + a_nv_n$. Then show that $a_i = \dfrac{v^T v_i}{||v_i||^2}$. [**2 points**]
5. If $w_1, w_2, \ldots, w_\ell$ span the vector space W and $v_1, v_2, \cdots, v_k$ span the vector space V, then prove that V and W are orthogonal subspaces if and only if $w_i^T v_j = 0$ for all $i = 1, 2, \ldots, \ell$ and $j = 1, 2, \ldots, k$. [**4 points**]
6. Find an orthonormal basis for the subspace W of $\mathbb{R}^4$ consisting of all vectors that are orthogonal to the given vector $(1, 2, -1, -3)^T$. [**4 points**]
7. Prove that a matrix Q is orthogonal if and only if its columns form an orthonormal basis. Prove that the transpose of an orthogonal matrix is also orthogonal. Prove that the inverse of an orthogonal matrix is orthogonal. [**3+2+2 points**]