MA1011: Problem Sheet 9 (Continuity)

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This problem sheet will NOT be graded, so you do not have to submit this to me. We will discuss the problems in the tutorials.

1. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$ and let x_0 be a cluster point of $A \cap B$. If $\lim_{x \to x_0} f(x) = L$ and $\lim_{x \to x_0} g(x) = T$, then using the definition of limits, prove that

   (a) $\lim_{x \to x_0} (f(x) + g(x)) = L + T$, and

   (b) $\lim_{x \to x_0} (f(x)g(x)) = LT$.

2. Show that the following inequality is true for $x \in (-\pi/2, \pi/2), x \neq 0$:

$$\cos x < \frac{\sin x}{x} < 1.$$

3. Show that the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

   is discontinuous at $x_0 = 0$.

4. Show that the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x \sin \dfrac{1}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases},$$

   is continuous at $x_0 = 0$.

5. Prove the Weierstraß Extreme Value Theorem.