Supremum and Infimum:

Defn: Given a fn. $f: A \to B$, we say that the fn is:

- bounded from above if the image $f(A)$ is bounded from above i.e. $\exists\, M \in \mathbb{R}$ s.t. $f(x) \leq M$, $\forall\, x \in A$.
- similarly defns for bounded from below, bounded or unbounded.

eg: $f(x) = x^2$ is bounded from below.

$g(x) = -x^2$ - - - - - above.

$\sin x$ is bounded from above as well as below.

Defn: Given a set $S \subset \mathbb{R}$,

(1) The supremum of S, denoted by $\sup S$, is the real no defined as the least upper bound (l.u.b.) of the set S.

i.e. (i) $x \leq \sup S$ $\forall\, x \in S$,

(ii) $\forall\, \varepsilon > 0$, $\exists\, x \in S$ s.t. $x > \sup S - \varepsilon$.

(2) The infimum of S, denoted by $\inf S$, is the real no defined as the biggest upper bound / greatest upper bound (bub/gub) of the set S.

i.e. (i) $x \geq \inf S$ $\forall\, x \in S$,

(ii) $\forall\, \varepsilon > 0$, $\exists\, x \in S$ s.t. $x < \inf S + \varepsilon$.

Defn: Given a fn $f: A \to B$,

(1) If f is bounded from above, the supremum of the fn, denoted by $\sup_A f$, is defined to be the supremum of the image $f(A)$.

(2) If f is not bounded from above, $\sup_A f = +\infty$.

(3) If f is bounded from below, the infimum of the fn, denoted by $\inf_A f$, is defined to be the infimum of the image $f(A)$.

(4) If f is not bounded from below, $\inf_A f = -\infty$.

- $\inf_A f \le f(x) \le \sup_A f \quad \forall x \in A$.
- $\inf_A f = \sup_A f$ iff $f(x)$ is constant (use the previous point.)
- For $A' \subset A$ and $f : A \to B$ we have,

  $\inf_A f \le \inf_{A'} f \le \sup_{A'} f \le \sup_A f$.

  eg: $f(x) = x^2$, $A = [0,3]$, $A' = [1,2]$.

Defn: (1) If $\sup_A f \in f(A)$, we say that the fn f admits a maximum in the domain A and $\sup_A f = \max_A f$.

(2) If $\inf_A f \in f(A)$, we say that the fn f admits a minimum in the domain A and $\inf_A f = \min_A f$.

Defn: Any pt. x of the domain s.t. $f(x) = \max_A f$ is called max. pt. and any pt. x of the domain s.t. $f(x) = \min_A f$ is called min pt.

- Maximum pts. need not be unique, $f(x) = x^2$, $A = [-2,2]$.
- Max or min might not exist even if the fn is bounded.

eg: $f(x) = x^2$, $A = (-1, 1)$.