Double Integrals:

The double integral of a nonnegative fn $f(x,y)$ defined on a region in the plane is associated with the vol^{m} of the region under the graph of $f(x,y)$.

Let $Q = [a,b] \times [c,d]$, $f: Q \to \mathbb{R}$ be bounded.

Let P_1 and P_2 be partitions of $[a,b]$ and $[c,d]$ resp,

and suppose $P_1 = \{x_0, x_1, \dots, x_n\}$ and $P_2 = \{y_0, y_1, \dots, y_m\}$.

The partition $P = P_1 \times P_2$ decomposes Q into mn subrectangles.

Define $m_{ij} = \inf \{ f(x,y) \mid (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j] \}$.

$$L(P,f) = \sum_{i=1}^{n} \sum_{j=1}^{m} m_{ij} \, \Delta y_j \, \Delta x_i .$$

$$M_{ij} = \sup \{ f(x,y) \mid (x,y) \in [x_{i-1}, x_i] \times [y_{j-1}, y_j] \}$$

$$U(P,f) = \sum_{i=1}^{m} \sum_{j=1}^{n} M_{ij} \, \Delta y_j \, \Delta x_i .$$

Def^n: We say that $f(x,y)$ is integrable if both the upper and lower integral of $f(x,y)$ are equal.

OR, equivalently if $\sup L(P,f) = \inf U(P,f)$.

We denote the integral by $\iint_Q f(x,y)\,dx\,dy$ or $\iint_Q f(x,y)\,dA$.

Theorem: If a fn $f(x,y)$ is continuous on $Q = [a,b] \times [c,d]$ then f is integrable on Q.

Fubini's Theorem: Let $f: Q := [a,b] \times [c,d] \to \mathbb{R}$ be cont.

Then, $$\iint_Q f(x,y)\,dx\,dy = \int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_a^b \left(\int_c^d f(x,y)\,dy \right) dx.$$

Let $f(x,y) > 0$ $\forall (x,y) \in Q$ and f cont. Consider the solid S enclosed by Q, the planes $x=a$, $x=b$, $y=c$, $y=d$ and the surface $z = f(x,y)$. Then, $\iint_Q f(x,y)\,dx\,dy$ is the vol^m of S (almost).

For every $y \in [c,d]$ $A(y) = \int_a^b f(x,y)\,dx$ is the area of the cross-section of the solid S cut by a plane $\parallel$ to xz-plane. So, $\int_c^d \left(\int_a^b f(x,y)\,dx \right) dy = \int_c^d A(y)\,dy$ is the vol^m of the solid S, same for the other way.

<u>Properties:</u>

- $\iint_{R_1 \cup R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$.
- $\iint_R k f(x,y)\,dA = k \iint_R f(x,y)\,dA$, $k \in \mathbb{R}$
- $\iint_R [f(x,y) \pm g(x,y)]\,dA = \iint_R f(x,y)\,dA \pm \iint_R g(x,y)\,dA$.
- $\iint_R f(x,y)\,dA \geq 0$ if $f(x,y) \geq 0$ on R.
- $\iint_R f(x,y)\,dA \geq \iint_R g(x,y)\,dA$ if $f(x,y) \geq g(x,y)$ on R.

<u>Ex:</u> Calculate the volume under the plane $z = 4-x-y$ over the region R, where $0 \leq x \leq 2$, $0 \leq y \leq 1$ in the xy-plane.

<u>Sol^n:</u> ~~let~~ The vol^m is $\int_0^2 A(x)\,dx$, $A(x) = \int_0^1 (4-x-y)\,dy$.

So, $\int_0^2 A(x)\,dx = \int_0^2 \int_0^1 (4-x-y)\,dy\,dx = \int_0^2 \left(4y - xy - \frac{y^2}{2}\right)_0^1 dx$

$= 5$.//

eg: $\int_{-1}^{1}\int_{0}^{2} (1-6x^2y)\,dx\,dy = \int_{-1}^{1} \left[x-2x^3y\right]_0^2 dy = \int_{-1}^{1}(2-16y)\,dy = 4.$ //

<u>Double integral over general bounded regions</u>: $f(x,y)$ is a bdd fn, defined on a bdd region D in the plane. Let Q be a rectangle s.t. $D \subseteq Q$. Define $\tilde{f}(x,y)$ on Q as,

$$\tilde{f}(x,y) = \begin{cases} f(x,y) & , (x,y) \in D \\ 0 & , (x,y) \in Q \setminus D. \end{cases}$$

If $\tilde{f}(x,y)$ is int. over Q we say that $f(x,y)$ is int. over D, and $\iint_D f(x,y)\,dx\,dy = \iint_Q \tilde{f}(x,y)\,dx\,dy$.

There is no general method of evaluating this for a general D.

<u>Fubini's Theorem</u>: Let $f(x,y)$ be a bdd fn over a region D.

(1) If $D = \{(x,y) \mid a \leq x \leq b \text{ and } f_1(x) \leq y \leq f_2(x)\}$ for some cont. fn $f_1, f_2 : [a,b] \to \mathbb{R}$, then

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left(\int_{f_1(x)}^{f_2(x)} f(x,y)\,dy \right) dx.$$

(2) If $D = \{(x,y) \mid c \leq y \leq d \text{ and } g_1(y) \leq x \leq g_2(y)\}$ for some cont. $g_1, g_2 : [c,d] \to \mathbb{R}$, then

$$\iint_D f(x,y)\,dx\,dy = \int_c^d \left(\int_{g_1(y)}^{g_2(y)} f(x,y)\,dx \right) dy.$$

- The properties of such integrals are the same as in the previous case.

Examples:

(1) $I = \iint_R \sin x / x \, dA$, R is the triangle bdd by x-axis, $y = x$ and $x = 1$.

Sol^n: $I = \int_0^1 \left(\int_0^x \sin x / x \, dy \right) dx = \int_0^1 \left(y \frac{\sin x}{x} \Big|_0^x \right) dx = \int_0^1 \sin x \, dx$

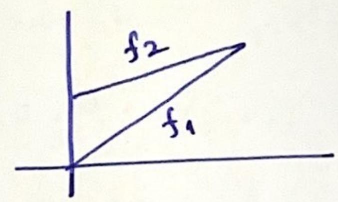
$= -\cos 1 + 1.$

(2) $I = \iint_D (x+y)^2 \, dx \, dy$, D is the region bdd by the lines joining $(0,0)$, $(0,1)$ and $(2,2)$

Use Fubini 1, $a = 0$, $b = 2$,

$f_1(x) = x$

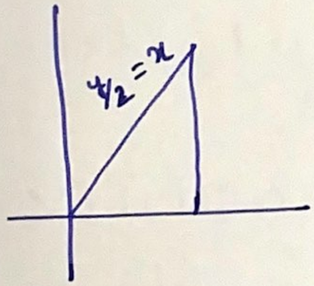
$f_2(x) = x/2 + 1$



$I = \int_0^2 \left(\int_x^{x/2+1} (x+y)^2 \, dy \right) dx.$

(3) $I = \int_0^2 \left(\int_{y/2}^1 e^{x^2} dx \right) dy. = \iint_D f(x,y) \, dx \, dy$, with

$D = \{ (x,y) \mid 0 \le y \le 2, \ y/2 \le x \le 1 \}$



By Fubini then, we have

$I = \int_0^1 \left(\int_0^{2x} e^{x^2} dy \right) dx = e - 1. //$

Area: The area of a closed, bdd planar region R is $\iint_R dA$.

eg: Find the area of the region R bdd by $y = x$, $y = x^2$ in the 1st quadrant.

$A = \int_0^1 \int_{x^2}^x dy \, dx = \int_0^1 y \Big|_{x^2}^x dx = \int_0^1 (x - x^2) dx = 1/6. //$

Change of Variables:

- We want something analogous to substitution for multiple integrals so that for regions where we cannot directly apply Fubini's theorem we transform them into ones where we can.
- We want to transform $\iint_S f(x,y)\,dx\,dy$ over a region S in the xy-plane to $\iint_T F(x,y)\,du\,dv$, defined on a new region T in the uv-plane.

For single integral we had $\int_a^b f(x)dx = \int_{g(a)}^{g(b)} f(g(t))\,g'(t)\,dt$.

Here we need two fns, $x = X(u,v)$, $y = Y(u,v)$.

- We assume that the mapping from T to S is one-one.
- The fns X and Y are cont. and have cont. partial derivatives, $\frac{\partial X}{\partial u}$, $\frac{\partial X}{\partial v}$, $\frac{\partial Y}{\partial u}$ and $\frac{\partial Y}{\partial v}$.
- The Jacobian $J(u,v) = \begin{vmatrix} \partial X/\partial u & \partial Y/\partial u \\ \partial X/\partial v & \partial Y/\partial v \end{vmatrix} \neq 0$.

Then, $$\iint_S f(x,y)\,dx\,dy = \iint_T f(X(u,v), Y(u,v))\,|J(u,v)|\,du\,dv.$$

eg: Find the area of the region S bounded by the hyperbolas $xy = 1$ and $xy = 2$ and the curves $xy^2 = 3$ and $xy^2 = 4$.

The reqd. area is $\iint_S dx\,dy$.

Put $u = xy$, $v = xy^2$. $\Rightarrow x = u^2/v$, $y = v/u$.

The region T is $1 \leq u \leq 2$ and $3 \leq v \leq 4$. The Jacobian $= 1/v$.

So, we get the area as $\iint_T \frac{1}{v}\,du\,dv = \int_3^4 \int_1^2 \frac{1}{v}\,du\,dv$. //

lar Coordinates:

We use the eq^{n}s $x = X(r,\theta) = r\cos\theta$, $y = Y(r,\theta) = r\sin\theta$.
We assume $r > 0$, $\theta \in [0, 2\pi)$ or $\theta_0 \le \theta < \theta_0 + 2\pi$ for some θ_0 so that the mapping is 1-1.

$$\therefore J(u,v) = \begin{vmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{vmatrix} = r.$$

$$\Rightarrow \iint_S f(x,y)\,dx\,dy = \iint_T f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta.$$

eg: Find the vol^m of the sphere of radius a.

- The vol^m is $V = 2\iint_S \sqrt{a^2 - x^2 - y^2}\,dx\,dy$, with $S = \{(x,y) \mid x^2 + y^2 \le a^2\}$.

In polar co-ordinates we get,

$$V = 2\iint_T \sqrt{a^2 - r^2}\, r\,dr\,d\theta, \text{ with } T = [0,a]\times[0,2\pi].$$

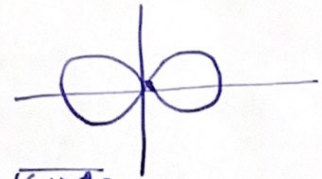
$$= 2\int_0^a \int_0^{2\pi} \sqrt{a^2 - r^2}\, r\,dr\,d\theta = 4\pi \int_0^a r\sqrt{a^2 - r^2}\,dr$$

$$= 4\pi \left[\frac{(a^2 - r^2)^{3/2}}{-3}\right]_0^a = \frac{4\pi a^3}{3}\ //.$$

Area in polar coordinates: Area of a closed and bounded region R is given by $A = \iint_R r\,dr\,d\theta$.

eg: Find the area enclosed by $r^2 = 4\cos 2\theta$.

- This is a lemniscate, so the area is 4 times the area in the 1st quadrant.



$$\text{So, } A = 4\int_0^{\pi/4}\int_0^{\sqrt{4\cos 2\theta}} r\,dr\,d\theta = 4\int_0^{\pi/4} \left[\frac{r^2}{2}\right]_0^{\sqrt{4\cos 2\theta}} d\theta$$

$$= 4\int_0^{\pi/4} 2\cos 2\theta\,d\theta = 4\left[\sin 2\theta\right]_0^{\pi/4} = 4\ //.$$

Ex: Evaluate $\iint\limits_R e^{x^2+y^2}\,dy\,dx$, where R is the semi-circle bounded by $y=0$ and $y=\sqrt{1-x^2}$.

Sol^n: $x = r\cos\theta,\ y = r\sin\theta$

$$\iint\limits_R e^{x^2+y^2}\,dy\,dx = \int_0^{\pi}\int_0^1 e^{r^2} r\,dr\,d\theta = \int_0^{\pi}\left[\frac{e^{r^2}}{2}\right]_0^1 d\theta$$

$$= \int_0^{\pi}\frac{1}{2}(e-1)\,d\theta = \pi/2\,(e-1).\ //$$

Ex: $\displaystyle\int_0^1\int_0^{1-x} e^{\frac{x-y}{x+y}}\,dx\,dy = I.$

Let $u = x-y,\ v = x+y$. Then, $I = \iint\limits_S e^{u/v}|J|\,du\,dv$,

where S is the region in the uv-plane bounded by $u=-v,\ u=v$ and $v=1$. //

$$I = \frac{1}{2}\int_0^1\int_{-v}^{v} e^{u/v}\,du\,dv = \frac{1}{2}\sinh(1).$$

Ex: Evaluate $\iint\limits_R \cos(9x^2+4y^2)\,dx\,dy$, where $R = \{(x,y)\mid 9x^2+4y^2 \le 1\}$.

Sol^n: Let $x = r/3\cos\theta,\ y = r/2\sin\theta$. Then $|J| = r/6$.

$$\iint\limits_R \cos(9x^2+4y^2)\,dx\,dy = \int_0^{2\pi}\int_0^1 \cos(r^2)\,r/6\,dr\,d\theta = \int_0^{2\pi}\int_0^1 \cos u\,\frac{du}{12}\,d\theta.\ //$$
