Directional derivatives, Gradient, Tangent Plane:

- Partial derivatives measures the rate of change of the fn along the direction of e_i. This notion is generalized now.

Def^n^: Let $f: \mathbb{R}^3 \to \mathbb{R}$, $X_0 \in \mathbb{R}^3$ and $U \in \mathbb{R}^3$ s.t. $\|U\| = 1$.
The directional derivative of f in the dir^n^ of U at $X_0 = (x_0, y_0, z_0)$ is defined by

$$D_{X_0} f(U) = \lim_{t \to 0} \frac{f(X_0 + tU) - f(X_0)}{t},$$

provided the limit exists.

- $D_{X_0} f(e_1) = f_x(X_0)$, $D_{X_0} f(e_2) = f_y(X_0)$, $D_{X_0} f(e_3) = f_z(X_0)$ etc.

Theorem: If f is diff. at X_0, then $D_{X_0} f(U)$ exists $\forall\, U \in \mathbb{R}^3$, $\|U\| = 1$. We also have, $D_{X_0} f(U) = f'(X_0) \cdot U$

$$= (f_x(X_0), f_y(X_0), f_z(X_0)) \cdot U.$$

- So if a fn is diff^n^ then all its dir^n^ derivatives exists and are easily computed.
- A fn may not be diff^n^ at a pt. but the dir^n^ derivatives in all dir^n^ may exist at that pt.

eg: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \begin{cases} \dfrac{x^2 y}{x^4 + y^2}, & (x,y) \neq (0,0) \\ 0, & \text{o/w}. \end{cases}$

f is not cont. at $(0,0)$ and hence not diff. at $(0,0)$.
Let $U = (u_1, u_2) \in \mathbb{R}^3$, $\|U\| = 1$, $\bar{0} = (0,0)$. Then we have,

$$\lim_{t \to 0} \frac{f(\bar{0} + tU) - f(\bar{0})}{t} = \lim_{t \to 0} \frac{t^3 u_1^2 u_2}{t(t^4 u_1^4 + t^2 u_2^2)} = \begin{cases} 0 & \text{if } u_2 = 0 \\ \dfrac{u_1^2}{u_2} & \text{if } u_2 \neq 0. \end{cases}$$

So, $D_{\bar{0}} f(u_1, 0) = 0$, $D_{\bar{0}} f(u_1, u_2) = \frac{u_1^2}{u_2}$, $u_2 \neq 0$.

That is, the dir^n derivatives in all dir^n at $(0,0)$ exists. //.

- Dir^n derivatives at a pt. w.r.t. one vector may exist but w.r.t. another might not.

eg: $f(x,y) = \begin{cases} x/y, & y \neq 0 \\ 0, & y = 0 \end{cases}$. Let $u = (u_1, u_2)$, $\|u\| = 1$.

Show that if $u_1 = 0$ or $u_2 = 0$ then $D_{\bar{0}} f(u)$ exists (like before).

If $u_1 u_2 \neq 0$ then we have,

$$\lim_{t \to 0} \frac{f(\bar{0} + tu) - f(\bar{0})}{t} = \lim_{t \to 0} \frac{u_1}{t u_2} \quad \text{which doesn't exist.}$$

eg: $f(x,y) = \begin{cases} \frac{y}{|y|} \sqrt{x^2 + y^2}, & y \neq 0 \\ 0, & y = 0 \end{cases}$

we have, $|f(x,y) - f(0,0)| = \sqrt{x^2 + y^2}$, so f is cont. at $(0,0)$.

Let $u = (u_1, u_2)$, $\|u\| = 1$. Then, $\lim\limits_{t \to 0} \frac{f(\bar{0} + tu) - f(\bar{0})}{t}$

$$= \lim_{t \to 0} \frac{f(tu_1, tu_2)}{t} = \begin{cases} 0, & \text{if } u_2 = 0 \\ \frac{u_2}{|u_2|}, & \text{if } u_2 \neq 0. \end{cases}$$

So, the dir^n derivatives at all pts. exist.

$f_x(0,0) = 0$, $f_y(0,0) = 1$

If f is diff at $\bar{0}$ then $f'(\bar{0}) = \alpha = (0,1)$.

$$\varepsilon(h,k) = \frac{\frac{k}{|k|}\sqrt{h^2 + k^2} - k}{\sqrt{h^2 + k^2}} \not\to 0 \text{ as } (h,k) \to (0,0)$$

(Take $h = k$)

So, f is not diff. at $(0,0)$. //.

<u>Def^n:</u> The vector $(f_x(X_0), f_y(X_0), f_z(X_0))$ is called the <u>gradient</u> of f at X_0 and is denoted by $\nabla f(X_0)$.

- If f is diff^n at X_0 then $f'(X_0) = \nabla f(X_0)$,

$$D_{X_0} f(U) = \nabla f(X_0) \cdot U$$
$$= \|\nabla f(X_0)\| \cos\theta$$

$\Rightarrow$ $D_{X_0} f(U)$ is max^m when $\theta = 0$ $(\nabla f(X_0) \neq 0)$.
min^m when $\theta = \pi$.

$\Rightarrow$ f increases (resp. decreases) most around X_0 in the dir^n $U = \dfrac{\nabla f(X_0)}{\|\nabla f(X_0)\|}$ $\left(\text{resp, } \dfrac{-\nabla f(X_0)}{\|\nabla f(X_0)\|}\right)$.

<u>Tangent Plane</u>: Let $f: \mathbb{R}^3 \to \mathbb{R}$ be diff^n and $c \in \mathbb{R}$. Consider the surface $S = \{(x,y,z) : f(x,y,z) = c\}$, called ~~the~~ a <u>level surface at height c</u>. Let $P = (x_0, y_0, z_0)$ be a pt on S and $R(t) = (x(t), y(t), z(t))$ be a diff^n curve lying on S. If T is the tangent vector to $R(t)$ at P then $\nabla f(P) \cdot T = 0$.
[$\because$ $R(t)$ lies on S, $f(x(t), y(t), z(t)) = c$, $\Rightarrow \dfrac{df}{dt} = 0$. By the chain rule, $\dfrac{\partial f}{\partial x} \cdot \dfrac{dx}{dt} + \dfrac{\partial f}{\partial y} \cdot \dfrac{dy}{dt} + \dfrac{\partial f}{\partial z} \cdot \dfrac{dz}{dt} = 0 \Rightarrow \nabla f \cdot \dfrac{dR}{dt} = 0$]

So, $\nabla f(P) \perp$ the tangent vector to every diff^n curve $R(t)$ on S passing through P. Thus, all these tangent vectors lie on a plane which is $\perp^r$ to $\nabla f(P)$. So, when $\nabla f(P) \neq 0$, the $\nabla f(P)$ is the normal to the surface of P.

The plane through P with normal $\nabla f(P)$ is defined by
$$f_x(P)(x - x_0) + f_y(P)(y - y_0) + f_z(P)(z - z_0) = 0.$$

This is called the <u>tangent plane</u> to the surface S at $P = (x_0, y_0, z_0)$.

If ~~the~~ $S = \{(x,y,f(x,y)) : (x,y) \in D \subseteq \mathbb{R}^2\}$, a graph of $f(x,y)$.

We can consider this as a level surface,

$$S = \{(x,y,z) : F(x,y,z) = 0\} \text{ where } F(x,y,z) = f(x,y) - z.$$

Let $X_0 = (x_0, y_0)$, $z_0 = f(x_0,y_0)$, $P = (x_0, y_0, z_0)$.

Then the eq^n of the tangent plane is,

$$\overset{F_x}{f_x(x_0,y_0)}(x - x_0) + \overset{F_y}{f_y(x_0,y_0)}(y-y_0) - \overset{F_z}{(z-z_0)} = 0$$

$$\Leftrightarrow \quad z = f(X) + f'(X_0)(X - X_0), \quad X = (x,y) \in \mathbb{R}^2. \text{//}$$

eg: Find the derivative of $f(x,y) = x^2 + xy$ at $(1,2)$ in the dir^n of $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

$$\lim_{t \to 0} \frac{f(1 + t/\sqrt{2},\ 2 + t/\sqrt{2}) - f(1,2)}{t} = \frac{5}{\sqrt{2}}. \text{//}$$

Q. Find the dir^n in which $f(x,y) = x^2/2 + y^2/2$, increases most rapidly and what are the ~~changes of~~ dir^n of zero change in f at $(1,1)$?

Soln: $(\nabla f)_{(1,1)} = (x,y)_{(1,1)} = (1,1)$.

The dir^n in which f increases most rapidly is in the dir^n of $(\nabla f)_{(1,1)}$ i.e. $\dfrac{(x,y)_{(1,1)}}{\|(x,y)\|_{(1,1)}} = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

The dir^n of zero change are the dir^n orthogonal to ∇f. //

Q: Find the eq^n for the tangent to the ellipse $x^2/4 + y^2 = 2$ at $(-2,1)$.

Sol^n: The ellipse is a level curve of $f(x,y) = x^2/4 + y^2$.

$(\nabla f)_{(-2,1)} = (-1,2)$. So, the tangent is the line,

$$-1(x+2) + 2(y-1) = 0 \text{ //.}$$

Q: Find the plane tangent to the surface $z = x\cos y - ye^x$ at $(0,0,0)$.

Sol^n: $f(x,y) = x\cos y - ye^x$, $f_x(0,0) = 1 = -f_y(0,0)$.

The tangent plane is, $(x-0) - 1(y-0) - (z-0) = 0$ //.

Algebra of gradients:

$$\nabla(kf) = k\nabla f, \quad k \in \mathbb{R}$$

$$\nabla(f \pm g) = \nabla f \pm \nabla g$$

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\nabla(f/g) = (g\nabla f - f\nabla g)/g^2.$$
