Functions of Several Variables:

Limit and Continuity:

- We say that L is the <u>limit</u> of a fn. $f: \mathbb{R}^3 \to \mathbb{R}$ at $X_0 \in \mathbb{R}^3$ and we write $\lim_{X \to X_0} f(X) = L$ if $f(X_n) \to L$ whenever a seq^{n} (X_n) in $\mathbb{R}^3$ with $X_n \neq X_0$, converges to X_0.
- A fn $f: \mathbb{R}^3 \to \mathbb{R}$ is continuous at $X_0 \in \mathbb{R}^3$ if $\lim_{X \to X_0} f(X) = f(X_0)$.

eg: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$, $(x,y) \neq (0,0)$ and $f(0,0) = 0$.

The fn is cont. at $(0,0)$ since $\left| \dfrac{xy}{\sqrt{x^2+y^2}} - 0 \right| \leq \dfrac{|x^2+y^2|}{\sqrt{x^2+y^2}}$

$= \sqrt{x^2+y^2} \to 0$ as $(x,y) \to (0,0)$.

eg: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = \dfrac{x^2 y}{x^4+y^2}$, when $(x,y) \neq (0,0)$,

$f(0,0) = 0$.

f is not cont. at $(0,0)$. One way to see this is,

$f(x, x^2) = \dfrac{x^4}{2x^4} = \dfrac{1}{2}$, so $f(x, x^2) \to 1/2$ as $x \to 0$.

<u>Partial derivatives:</u> The partial derivative of f. w.r.t. to the first variable at $X_0 = (x_0, y_0, z_0)$ is defined by

$$\left.\frac{\partial f}{\partial x}\right|_{X_0} = \lim_{h \to 0} \frac{f(x_0+h, y_0, z_0) - f(x_0, y_0, z_0)}{h}$$

provided the limit exists.

Similarly the other partial derivatives are defined.

eg: $f(x,y) = \dfrac{2xy}{x^2+y^2}$ at $(x,y) \neq (0,0)$, $f(0,0) = 0$.

$\dfrac{\partial f}{\partial x} = ?$ $\dfrac{\partial f}{\partial y} = ?$.

- The function might not be continuous at a point but the partial derivative may exist.

eg. The previous fn is not cont. at $(0,0)$.

Notice, $f(x, mx) = \dfrac{2mx^2}{~~2~~m^2x^2 + x^2} = \dfrac{2m}{1+m^2}$.

So, $f(x, mx) \to \dfrac{2m}{1+m^2}$ as $x \to 0$, hence the limit doesn't exist at ~~x = (x,y)~~ $(0,0)$.

Proposition: Let $f(x,y)$ be defined in $S = \{(x,y) \in \mathbb{R}^2 : a<x<b, c<y<d\}$. ~~Let~~ The partial derivatives of f exist and are bounded in S. Then, the function f is continuous in S.

Proof: Let $|f_x(x,y)| \leq M$, $|f_y(x,y)| \leq M$ $\forall (x,y) \in S$.

Now, $f(x+h, y+k) - f(x,y) = f(x+h, y+k) - f(x+h, y) + f(x+h, y) - f(x,y)$

$= k f_y(x+h, y+c_1k) + h f_x(x+c_2h, y)$ [for some $c_1, c_2 \in \mathbb{R}$ by MVT].

Therefore, $|f(x+h, y+k) - f(x,y)| \leq M(|h|+|k|)$

$\leq 2M\sqrt{h^2+k^2}$

So, for $\varepsilon > 0$ we choose $\delta = \dfrac{\varepsilon}{2M}$ and we are done. //.

Differentiability: Let $f: \mathbb{R}^3 \to \mathbb{R}$ and $X = (x_1, x_2, x_3)$. We say that f is differentiable at X if there exists an $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ in $\mathbb{R}^3$ such that the error function $\varepsilon(H) = \dfrac{f(X+H) - f(X) - \alpha \cdot H}{\|H\|}$ tends to 0 as $H \to 0$.

We then write $f'(X) = (\alpha_1, \alpha_2, \alpha_3)$.

This can be reconciled with differentiability for a real-valued fns as well, as we can say a fn $f:\mathbb{R}\to\mathbb{R}$ is diff. at x iff there exists $\alpha\in\mathbb{R}$ s.t.

$$\frac{|f(x+h)-f(x)-\alpha.h|}{|h|}\to 0 \text{ as } h\to 0.$$

<u>Theorem</u>: Let $f:\mathbb{R}^3\to\mathbb{R}$, $X\in\mathbb{R}^3$. If f is differentiable at X then f is continuous at X.

<u>Proof</u>: Let f be differentiable at X, then $\exists\ \alpha=(\alpha_1,\alpha_2,\alpha_3)\in\mathbb{R}^3$ s.t. $|f(X+H)-f(X)-\alpha\cdot H| = \|H\|\,\mathcal{E}(H)$ and $\mathcal{E}(H)\to 0$ as $H\to 0$.

$$\Rightarrow |f(X+H)-f(X)| \leq \|H\|\,\mathcal{E}(H) + \|H\|\,(|\alpha_1|+|\alpha_2|+|\alpha_3|)$$

and $\mathcal{E}(H)\to 0$ as $H\to 0$.

$\Rightarrow f(X+H)\to f(X)$ as $H\to 0$ which proves that f is continuous at X. //

<u>Theorem</u>: Suppose f is differentiable at X. Then the partial derivatives $\left.\frac{\partial f}{\partial x}\right|_X$, $\left.\frac{\partial f}{\partial y}\right|_X$ and $\left.\frac{\partial f}{\partial z}\right|_X$ exist and we have

$$f'(X)=\left(\left.\frac{\partial f}{\partial x}\right|_X, \left.\frac{\partial f}{\partial y}\right|_X, \left.\frac{\partial f}{\partial z}\right|_X\right).$$

<u>Proof</u>: Suppose f is diff. at X and $f'(X)=(\alpha_1,\alpha_2,\alpha_3)$. We take $H=(t,0,0)$ to get, $\mathcal{E}(H)=\frac{f(X+H)-f(X)-\alpha_1 t}{|t|}\to 0$ as $t\to 0$

$$\Rightarrow \alpha_1=\left.\frac{\partial f}{\partial x}\right|_X. \ //$$

<u>eg</u>: $f(x,y)=xy\frac{x^2-y^2}{x^2+y^2}$ at $(x,y)\neq(0,0)$, $f(0,0)=0$.

To verify that f is diff. at $(0,0)$ we choose $\alpha=\left.\left(\frac{\partial f}{\partial x},\frac{\partial f}{\partial y}\right)\right|_{(0,0)}$ and then check $\mathcal{E}(H)\to 0$ as $H=(h,k)\to 0$.

We get $\alpha = (0,0)$ in this case.

$$|\varepsilon(H)| = \frac{|f(0+H) - f(0) - (0,0)\cdot H|}{\|H\|} \leq \left| \frac{hk}{\sqrt{h^2+k^2}} \right| \leq \sqrt{h^2+k^2} \to 0$$

as $H \to 0$.

So, f is diff. at $(0,0)$.

- Partial derivatives may exist, the fn may be continuous but not be differentiable.

eg: $$f(x,y) = \begin{cases} \dfrac{2x^2y + y^3}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

- The fn is cont. at $(0,0)$.
- The partial derivaties exist, $\dfrac{\partial f}{\partial x} = \dfrac{2y^3 x}{(x^2+y^2)^2}$

$$\frac{\partial f}{\partial y} = \frac{y^4 + y^2x^2 + 2x^4}{(x^2+y^2)^2}.$$

But the limits $\lim\limits_{\substack{x\to 0\\ y\to 0}} \dfrac{\partial f}{\partial x}$ & $\lim\limits_{(x,y)\to(0,0)} \partial f/\partial y$ don't exist.

<u>Theorem</u>: If $f: \mathbb{R}^3 \to \mathbb{R}$ is such that all its partial derivaties exist in a neighbourhood of X_0 and cont. at X_0 then f is diff. at X_0.

- The converse of this is NOT true.

eg: $$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\dfrac{1}{\sqrt{x^2+y^2}}\right) & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

$f(x,y)$ is diff. at $(0,0)$ but the partial derivatives are not continuous at $(0,0)$.

Increment Theorem: Let $f(x,y)$ be differentiable at (x_0,y_0), then we have

$$\Delta f = f(x_0+\Delta x,\, y_0+\Delta y) - f(x_0,y_0) = f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y,$$

where $\varepsilon_1(\Delta x,\Delta y),\ \varepsilon_2(\Delta x,\Delta y) \to 0$ as $\Delta x \to 0$, $\Delta y \to 0$.

Proof: Let $H = (\Delta x, \Delta y)$. Since the fn is diff. at (x_0,y_0) we have,

$$f(x_0+\Delta x,\, y_0+\Delta y) - f(x_0,y_0) = f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y + \|H\|\,\varepsilon(H),$$

$\varepsilon(H) \to 0$ as $H \to 0$.

We have,

$$\varepsilon(H)\,\|H\| = \frac{\varepsilon(H)}{\|H\|}(\Delta x^2 + \Delta y^2) = \Delta x\left(\frac{\varepsilon(H)}{\|H\|}\Delta x\right) + \Delta y\left(\frac{\varepsilon(H)}{\|H\|}\Delta y\right).$$

Define, $\varepsilon_1(H) = \Delta x \dfrac{\varepsilon(H)}{\|H\|}$, $\varepsilon_2(H) = \Delta y \dfrac{\varepsilon(H)}{\|H\|}$. //.

Chain Rule: Let $f(x,y)$ be diff and if $x = x(t)$, $y = y(t)$ are diff. fns on t then the fn $w = f(x(t), y(t))$ is diff at t and we have

$$\frac{df}{dt} = f_x(x(t), y(t))\,x'(t) + f_y(x(t), y(t))\,y'(t) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}.$$

Proof: By the increment theorem we have,

$$\Delta f = f_x(x_0,y_0)\,\Delta x + f_y(x_0,y_0)\,\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta x,$$

$\varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$

We have, $$\frac{\Delta f}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

Let $\Delta t \to 0 \Rightarrow \varepsilon_1, \varepsilon_2 \to 0$ as $\Delta x, \Delta y \to 0$ and we get,

$$\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}. //$$

Problems:

① $f(x,y) = x\cos y + ye^x$, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)$

(Euler) $f_{yx} = f_{xy}$

$$= \frac{\partial}{\partial y}(\cos y + ye^x)$$
$$= -\sin y + e^x. //$$

② Find $\frac{dw}{dt}$, $w = xy + z$, $x = \cos t$, $y = \sin t$, $z = t$.

By Chain rule we have,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt}$$
$$= y(-\sin t) + x\cos t + 1$$
$$= -\sin^2 t + \cos^2 t + 1 = 1 + \cos 2t. //$$

③ Find $\frac{\partial w}{\partial x}$ if $w = x^2 + y^2 + z^2$, $z^3 - xy + yz + y^3 = 1$ and x & y are independent variables.

$$\frac{\partial w}{\partial x} = 2x + 2z\frac{\partial z}{\partial x} \qquad \text{—— ①}$$

$$3z^2\frac{\partial z}{\partial x} - y + y\frac{\partial z}{\partial x} = 0 \qquad \text{—— ②}$$

$$\Rightarrow \frac{\partial z}{\partial x} = \frac{y}{y + 3z^2}$$

$$① \Rightarrow \frac{\partial w}{\partial x} = 2x + \frac{2yz}{y + 3z^2}. //$$