# Lagrange Multiplier Method:

• For $f: S \to \mathbb{R}$, $S \subset \mathbb{R}^3$, $X_0 \in S$ we can apply the first and second derivative tests to check if $X_0$ is a local max$^{\underline{m}}$ or min$^{\underline{m}}$ when $X_0$ is an interior point. If $X_0$ is NOT an interior pt we cannot apply these tests.

• For the case when the constrained set $S$ is a level surface, like a sphere we can use the Langrange multiplier method:

Suppose $f$ and $g$ have continuous partial derivatives. Let $(x_0, y_0, z_0) \in S := \{(x,y,z) \mid g(x,y,z) = 0\}$ and $\nabla g(x_0, y_0, z_0) \neq 0$. If $f$ has a local max$^{\underline{m}}$ or min$^{\underline{m}}$ at $(x_0, y_0, z_0)$ then there exists $\lambda \in \mathbb{R}$ s.t. $\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$.

• In practice we use the following eq$^{n}$s:

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \text{ and } g(x,y,z) = 0. \quad \text{——} \quad (A)$$

We solve for unknowns $x, y, z$ & $\lambda$. And the local extremum pts are found among the sol$^{\underline{n}}$s:

Examples:

(1) Find a point on the plane $2x + 3y - z = 5$ in $\mathbb{R}^3$ which is nearest to $(0,0,0)$.

We need to minimize $f(x,y,z) = x^2 + y^2 + z^2$ subject to the constraint $g(x,y,z) = 2x + 3y - z - 5 = 0$. Here $\nabla g \neq 0$.

(A) $\Rightarrow$ $2x = 2\lambda$, $2y = 3\lambda$, $2z = -\lambda$, $2x + 3y - z = 5$.

Solving we obtain $\lambda = 5/7$, $(x,y,z) = (5/7, 15/14, -5/14)$.

Since $f$ attains a min$^{\underline{m}}$ at this pt so, the pt is the reqd. nearest point.

• Here we used $f(x,y,z) = x^2+y^2+z^2$ because we wanted to find the min$^{\underline{m}}$ value of $\sqrt{(x-0)^2+(y-0)^2+(z-0)^2}$.

(2) $f(x,y) = 2-x^2-2y^2$ w.r.t. $g(x,y) = x^2+y^2-1$.

(A) $\Rightarrow$ $2x+2\lambda x = 0$, $4y+2\lambda y = 0$, $x^2+y^2-1=0$.

$\Rightarrow \lambda = -1, -2$. $\Rightarrow$ For $\lambda=-1$, $y=0$, $x=\pm 1$, $f(x,y)=1$

For $\lambda = -2$, $y=\pm 1$, $x=0$, $f(x,y)=0$.

Since the fn acheives the max$^{\underline{m}}$ and min$^{\underline{m}}$ over the closed and bounded set $x^2+y^2=1$, the pts $(0,\pm 1)$ & $(\pm 1, 0)$ are the minima and maxima resp.

• The condition $\nabla g(x_0,y_0,z_0) \neq 0$ cannot be droped, and a pt. where $\nabla g$ is $(0,0)$ cannot be an extremeum.

eg: Min $f(x,y) = x^2+y^2$ subject to $g(x,y) = (x-1)^3 - y^2 = 0$.
We want to find a pt. on the curve $y^2 = (x-1)^3$ which is nearest to the origin of $\mathbb{R}^2$. Geometrically this pt. should be $(1,0)$. Here $\nabla g(1,0) = 0$, $\nabla f(1,0) = (2,0)$ so we cannot have $\nabla f(1,0) = \lambda \nabla g(1,0)$ for any $\lambda \in \mathbb{R}$.

• The method works in $\mathbb{R}^n$ as well.

eg: $a_1, a_2, \dots, a_n \in \mathbb{R}^+$, max $a_1x_1+\dots+a_nx_n$ with $x_1^2+\dots+x_n^2=1$.
Here, $f(x_1,x_2,\dots,x_n) = a_1x_1+\dots+a_nx_n$, $g(x_1,\dots,x_n) = x_1^2+\dots+x_n^2-1$.

(A) $\Rightarrow$ $a_1 = 2\lambda x_1, \dots, a_n = 2\lambda x_n$, $x_1^2+\dots+x_n^2=1$.

$\Rightarrow a_1^2+\dots+a_n^2 = 4\lambda^2 \Rightarrow \lambda = \pm\dfrac{\sqrt{a_1^2+\dots+a_n^2}}{2}$.

Since the cont. fn $f$ has its min$^{\underline{m}}$ and max$^{\underline{m}}$ on the closed and bdd subset $x_1^2+\dots+x_n^2=1$, so, $\lambda = \dfrac{\sqrt{a_1^2+\dots+a_n^2}}{2}$ gives the max$^{\underline{m}}$ and $\lambda = -\dfrac{\sqrt{a_1^2+\dots+a_n^2}}{2}$ the min$^{\underline{m}}$.

• In case we have two constraints, $g_1(x,y,z)=0$, $g_2(x,y,z)=0$; with $g_1$ & $g_2$ diff and $\nabla g_1$ not parallel to $\nabla g_2$, then we introduce two Lagrange multipliers $\lambda$ and $\mu$. So, to locate the points $P(x,y,z)$ where $f$ takes on its constrained extreme values we find $x, y, z, \lambda$ & $\mu$ that satisfy,

$$\nabla f = \lambda \nabla g_1 + \mu \nabla g_2, \quad g_1 = 0, \quad g_2 = 0 \quad \text{——} \quad (B).$$

Geometry: • The surfaces $g_1=0$, $g_2=0$ intersect in a smooth curve, $C$.

• We want pts along $C$ where $f$ has local extremum, these pts are where $\nabla f$ is normal to $C$.

[ Othogonal Gradient Theorem: Suppose $f(x,y,z)$ is diff in a region whose interior contains a smooth curve $C = (g(t), h(t), k(t))$. If $P_0 \in C$ where $f$ has a local extremum then $\nabla f$ is $\perp^r$ to $C$ at $P_0$.

Proof: $\dfrac{df}{dt} = \dfrac{\partial f}{\partial x}\cdot\dfrac{dg}{dt} + \dfrac{\partial f}{\partial y}\cdot\dfrac{dh}{dt} + \dfrac{\partial f}{\partial z}\cdot\dfrac{dk}{dt} = \nabla f \cdot v$ where

$$v = \left(\frac{dg}{dt}, \frac{dh}{dt}, \frac{dk}{dt}\right).$$

At a pt $P_0$ where $f$ has local $ext^{\underline{m}}$ we have $\dfrac{df}{dt} = 0$ so,

$\nabla f \cdot v = 0$. // ].

• ~~These pts~~. We also have $\nabla g_1$, $\nabla g_2$ are normal to $C$ at these pts as $C$ lies in the surfaces $g_1 = 0$, $g_2 = 0$.

• So, $\nabla f$ lies in the plane determined by $\nabla g_1$, $\nabla g_2$, which means $\nabla f = \lambda \nabla g_1 + \mu \nabla g_2$.

• Since these pts also lie on the surfaces so these pts must also satisfy $g_1 = 0$, $g_2 = 0$. //

(This method can be extended for 'n' constraints as well.)

eg: The plane $x+y+z=1$ cuts the cylinder $x^2+y^2=1$ in an ellipse. Find the points on the ellipse that lie closest to and farthest from $(0,0,0)$.

Soln: $f(x,y,z) = x^2+y^2+z^2$ (sq. of distance from $(x,y,z)$ to $(0,0,0)$).

$g_1(x,y,z) = x^2+y^2-1 = 0$, $g_2(x,y,z) = x+y+z-1 = 0$.

Solving (B) we get, $2x = 2\lambda x + \mu$, $2y = 2\lambda y + \mu$, $2z = \mu$.

$$\Rightarrow z = \mu/2, \quad x = z/1-\lambda, \quad y = z/1-\lambda. \quad \text{——— } (*)$$

$(*)$ is satisfied if either $z=0$, $\lambda = 1$ or $\lambda \neq 1$, $x = y = z/1-\lambda$.

If $z=0$ then $g_1=0, g_2=0$ gives the pts $(1,0,0)$ & $(0,1,0)$.

If $x=y$ then $g_1=0, g_2=0$ gives $x = \pm\sqrt{2}/2$, $z = 1 \mp \sqrt{2}$. The points are $(\sqrt{2}/2, \sqrt{2}/2, 1-\sqrt{2})$ & $(-\sqrt{2}/2, -\sqrt{2}/2, 1+\sqrt{2})$.

The pts closest to the origin are $(1,0,0)$ & $(0,1,0)$.

The pt farthest from the origin is $(-\sqrt{2}/2, -\sqrt{2}/2, 1+\sqrt{2})$ //.

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