Line Integrals:

Let $R:[a,b] \to \mathbb{R}^3$ and C be a parametric curve defined by $R(t)$, that is $C(t) = \{R(t) \mid t \in [a,b]\}$.

Suppose $f: C \to \mathbb{R}^3$ is a bdd fn.

Defn: Suppose R is a diff. fn. The line integral of f over C is denoted by $\int_C f.dR$ and is defined as,

$$\int_C f.dR = \int_a^b f(R(t)).R'(t)\,dt$$

provided the integral on the R.H.S. exists.

- Suppose $f = (f_1, f_2, f_3)$ and $R(t) = (x(t), y(t), z(t))$

then $\int_C f.dR = \int_C f_1 dx + f_2 dy + f_3 dz = \int_C f_1(x,y,z)\,dx + f_2(x,y,z)\,dy + f_3(x,y,z)\,dz$.

eg: Compute the line integral from $(0,0,0)$ to $(1,2,4)$ if $f = (x^2, y, (xz-y))$, along the curve defined parametrically by $x = t^2, y = 2t, z = 4t^3$.

$$\int_C f.dR = \int_C x^2 dx + y\,dy + (xz - y)\,dz$$

$$= \int_0^1 t^2 dt + (2t)(2dt) + (4t^3.t^2 - 2t)(12t^2 dt).$$

eg: Evaluate $\int_C \frac{-y\,dx + x\,dy}{x^2 + y^2}$ where $C = \{(x,y) \mid x^2 + y^2 = r^2, r \neq 0\}$.

Let $C = (r\cos t, r\sin t)$, $0 \leq t \leq 2\pi$.

Then, $\int_C \frac{-y\,dx + x\,dy}{x^2 + y^2} = \int_0^{2\pi} \frac{\sin^2 t + \cos^2 t}{\sin^2 t + \cos^2 t}\,dt = 2\pi$ //.

- If C is made by joining a finite no. of curves $C_1, C_2, \dots, C_n$ end to end then,

$$\int_C f.dR = \int_{C_1} f.dR + \dots + \int_{C_n} f.dR.$$

The Second Fundamental Theorem of Calculus: If $f: [a,b] \to \mathbb{R}$, and f' is cont. then, $\int_a^b f'(t)dt = f(b) - f(a)$. (First FTC).

Let $S \subset \mathbb{R}^3$, $f: S \to \mathbb{R}$ be diff. on S and ∇f is cont. Let A, B be two pts. on S. Let $C = \{R(t) \mid t \in [a,b]\}$ be a curve lying in S and joining the pts A and B, that is $R(a) = A$, $R(b) = B$. Suppose $R'(t)$ is cont. on $[a,b]$.

Then, $$\int_C \nabla f . dR = f(B) - f(A).$$

Proof: Let $g(t) = f(R(t))$.

$$\int_C \nabla f . dR = \int_a^b \nabla f(R(t)). R'(t)\, dt$$
$$= \int_a^b g'(t)\, dt = g(b) - g(a)$$
$$= f(R(b)) - f(R(a)) = f(B) - f(A). //$$

Green's Theorem: Let C be a piecewise smooth simple closed curve in the xy-plane and let D be the closed region enclosed by C. Suppose M, N, N_x, M_y are real valued cont. fn in an open set containing D. Then,

$$\iint_D (N_x - M_y)\, dx\, dy = \oint_C M\, dx + N\, dy,$$

where the line integral is taken around C in the counterclockwise direction.

- This is a 2-D analog of the FTC.
- Let $R:[a,b]\to\mathbb{R}^3$ be cont. If $R(a)=R(b)$ then the curve described by R is <u>closed</u>. A closed curve such that $R(t_1)\neq R(t_2)$ $\forall\, t_1,t_2\in(a,b]$ is called a <u>simple closed curve</u>. If R' exists and is cont, then the curve described by R is called <u>smooth</u>. The curve is called <u>piecewise smooth</u> if $[a,b]$ can be partitioned into a finite no. of subintervals and in each of which the curve is smooth.

<u>Area expressed as a line integral</u>: Let C be a simple piecewise smooth closed curve and D be the region enclosed by C. Let $N(x,y)=x/2$, $M(x,y)=-y/2$, then by Green's theorem the area of D is,

$$a(D)=\iint_D dx\,dy=\iint_D (N_x-M_y)\,dx\,dy=\int_a^b M\,dx+N\,dy$$
$$=\frac{1}{2}\int_C -y\,dx+x\,dy.$$

eg: (1) Show that the value of $\int_C xy^2\,dx+(x^2y+2x)\,dy$ around any sq. depends only on the size of the sq. C and not on its location in the plane.

- Let R be a sq. enclosed by the boundary C.

By Green's theorem $\int_C xy^2\,dx+(x^2y+2x)\,dy=\iint_R 2\,dx\,dy$
$$=2\,\text{Area}(R)\,//.$$

(2) Find the area ~~of~~ bounded by the ellipse $C: x^2/a^2 + y^2/b^2 = 1$.

Parametrize C by $(a\cos t, b\sin t)$, $0 \le t \le 2\pi$.

The area is $\frac{1}{2}\int_C -y\,dx + x\,dy = \frac{1}{2}\int_0^{2\pi} -(b\sin t)(-a\sin t)\,dt + (a\cos t)(b\cos t)\,dt$

$$= \frac{1}{2}\int_0^{2\pi} ab\,dt = ab\pi. //$$

(3) Evaluate $\oint_C xy\,dy - y^2 dx$.

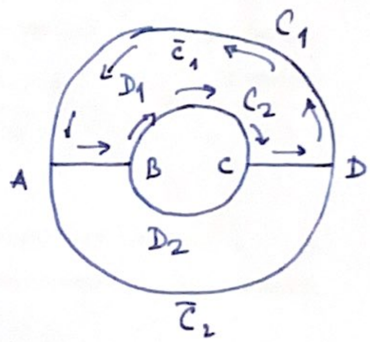
- Take $M = xy$, $N = y^2$, C and R as the sq's boundary and interior.

$$\oint_C xy\,dy - y^2 dx = \iint_R (y + 2y)\,dx\,dy = \int_0^1\int_0^1 3y\,dx\,dy = 3/2. //$$

Extended Green's Theorem: Suppose C_1 and C_2 be two circles.

- Let D be the annular region bet^n the two circles.

We introduce cuts AB and CD as shown in the figure. Consider the simple curve $\bar{C}_1$ which is the upper half of C_2, the upper half of C_1, the segments AB and CD. Similarly,



the closed curve $\bar{C}_2$ is the lower half of C_2, the lower half of C_1, the segments AB and CD. Let D_1 and D_2 be the regions enclosed by $\bar{C}_1$ and $\bar{C}_2$.

Let M and N be two continuously diff. scalar valued fns on an open set containing D. We can apply Green's theorem to each of D_1 and D_2 and add the quantities up. We then obtain,

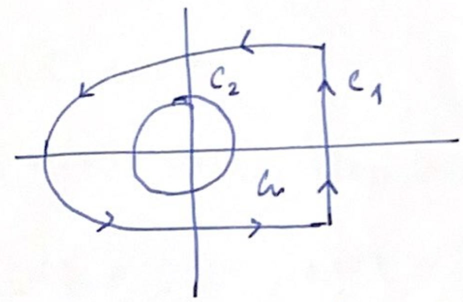
$$\iint_D (N_x - M_y)\,dx\,dy = \oint_{C_1} (M\,dx + N\,dy) - \oint_{C_2} (M\,dx + N\,dy).$$

(The line integrals along the cuts cancel out.).

The minus sign appears because the line integral on the part of $\overline{C}_2$ is taken along the clockwise direction.

This idea can be generalized to regions enclosed by two or more simple closed curves.

eg: Let G be the region outside the unit circle bdd on left by the parabola $y^2 = 2(x^2+2)$, on the right by the line $x=2$.



Evaluate $\int_{C_1} \frac{x\,dy - y\,dx}{x^2+y^2}$ where C_1 is the outer boundary of G oriented counterclockwise.

- Let C_2 be the unit circle. Let $M = \frac{-y}{x^2+y^2}$, $N = \frac{x}{x^2+y^2}$.

$N_x - M_y = 0$ here.

So, $\iint_G (N_x - M_y)\,dx\,dy = 0$.

By applying Green's theorem we get,

$$\oint_{C_1} (M\,dx + N\,dy) = \oint_{C_2} (M\,dx + N\,dy).$$

$$= 2\pi \quad \text{(From before.)}.$$