Parametric Surfaces:

Let T be a region in $\mathbb{R}^2$, and $r(u,v) = (X(u,v), Y(u,v), Z(u,v))$ be a cont. fn on T. The range of r, $\{ r(u,v) \mid (u,v) \in T \}$ is called a <u>parametric surface</u> with the parameter domain T and parameters u and v.

The map r is 1-1 in the interior of T, so there are no crossings. The surface $r(u,v)$ is also expressed as

$$x = X(u,v),\ y = Y(u,v),\ z = Z(u,v), \quad (u,v) \in T.$$

These are called the <u>parametric eq^{n}s</u> of the surface.

<u>eg:</u> (1) $a > 0$, $a \in \mathbb{R}$, $t \in \mathbb{R}$, $0 \le \theta \le 2\pi$, the eq^{n}s

$$x = a\cos\theta,\ y = a\sin\theta,\ z = t$$ represents a cylinder.

(2) $a > 0$, $a \in \mathbb{R}$, $0 \le \theta \le 2\pi$, $0 \le \phi \le \pi$, the eq^{n}s

$$x = a\sin\phi\cos\theta,\ y = a\sin\phi\sin\theta,\ z = a\cos\phi$$ is a sphere.

<u>Area:</u> Let $S = r(u,v)$ be a parametrized surface defined on a parameter domain T. Let r_u and r_v be cont. on T and $r_u \times r_v \neq 0$ on T.

(<u>Aside</u>: $r_u \times r_v = \|r_u\| \|r_v\| \sin\theta\, n$, where θ is the angle bet^n r_u and r_v in the plane containing them, and n is the unit vector $\perp^r$ to the plane containing r_u and r_v. Note, cross product being 0 means r_u and r_v are parallel.)

The area of S, denoted by $a(S)$ is defined as,

$$a(S) = \iint_T \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| du\,dv.$$

~~a~~ of a surface defined by a graph: Let S be a surface given by $z=f(x,y)$, $(x,y)\in T$. Then, S can be considered as a parametric surface defined by ②

$$r(x,y) = (x,\ y,\ f(x,y)),\quad (x,y)\in T.$$

Then, $$a(S) = \iint_T \sqrt{1+f_x^2+f_y^2}\,dx\,dy.$$

eg: (1) Find the area of the surface of the portion of the sphere $x^2+y^2+z^2 = 4a^2$ that lies inside the cylinder $x^2+y^2=2ax$.

The sphere can be considered as a union of two graphs $z = \pm\sqrt{4a^2-x^2-y^2}$. Let $z=f(x,y) = \sqrt{4a^2-x^2-y^2}$.

$$f_x = \frac{-x}{\sqrt{4a^2-x^2-y^2}},\quad f_y = \frac{-y}{\sqrt{4a^2-x^2-y^2}}$$

$$\sqrt{1+f_x^2+f_y^2} = \sqrt{\frac{4a^2}{4a^2-x^2-y^2}}.$$

Let T be the projection of the surface $z=f(x,y)$ on the xy-plane. So, by symmetry we have,

$$a(S) = 2\iint_T \sqrt{\frac{4a^2}{4a^2-x^2-y^2}}\,dx\,dy.$$

(Convert to polar coordinates and evaluate the integral.)

- We have, $$\|r_u\times r_v\|^2 = \|r_u\|^2\|r_v\|^2\sin^2\theta$$
$$= \|r_u\|^2\|r_v\|^2(1-\cos^2\theta)$$
$$= \|r_u\|^2\|r_v\|^2 - (r_u\cdot r_v)^2.$$

So, we can write, $a(S) = \iint_T \sqrt{EG-F^2}\,du\,dv$, where

$$E = r_u\cdot r_u,\quad G = r_v\cdot r_v,\quad F = r_u\cdot r_v.$$

(2) Find the area of the torus

$x = (a + b\cos\phi)\cos\theta, \quad y = (a + b\cos\phi)\sin\theta, \quad z = b\sin\phi,$

where $0 \le \theta \le 2\pi$, $0 \le \phi \le 2\pi$, $0 < b < a$, $a, b \in \mathbb{R}$.

$r_\theta = \left(-(a + b\cos\phi)\sin\theta, \ (a + b\cos\phi)\cos\theta, \ 0\right)$

$r_\phi = \left(-b\cos\phi\cos\theta, \ -b\sin\phi\sin\theta, \ b\cos\phi\right).$

$r_\theta \cdot r_\theta = (a + b\cos\phi)^2, \quad r_\phi \cdot r_\phi = b^2, \quad r_\theta \cdot r_\phi = 0.$

$\sqrt{EG - F^2} = b(a + b\cos\phi).$

$$\text{So, } a(S) = \iint_T (a + b\cos\phi)\, b \, d\theta \, d\phi = \int_0^{2\pi}\int_0^{2\pi} b(a + b\cos\phi)\, d\theta \, d\phi.$$

Surface Integrals: Let S be a parametric surface defined by $r(u,v)$, $(u,v) \in T$, and let r_u and r_v be cont. Let $g: S \to \mathbb{R}$ be bdd. The surface integral of g over S is denoted by $\iint_S g \, d\sigma$ and is defined as,

$$\iint_S g \, d\sigma = \iint_T g(r(u,v)) \, \|r_u \times r_v\| \, du \, dv$$

$$= \iint_T g(r(u,v)) \sqrt{EG - F^2} \, du \, dv,$$

provided the double integral on the R.H.S. exists.

If S is defined by $z = f(x,y)$, then we have,

$$\iint_S g \, d\sigma = \iint_T g(x, y, f(x,y)) \sqrt{1 + f_x^2 + f_y^2} \, dx \, dy,$$

where T is the projection of the sphere S over the xy-plane.

Q: (1) Let S be the hemispherical surface $z = (a^2 - x^2 - y^2)^{1/2}$. (4)

[Ev]aluate $\iint_S \frac{d\sigma}{(x^2+y^2+(z+a)^2)^{1/2}}$.

We parametrize S as, $S := r(\theta, \phi)$

$$= (a \sin\phi \cos\theta,\ b \sin\phi \cos\theta,\ a \cos\phi)$$

$$(0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi).$$

$\sqrt{EG - F^2} = a^2 \sin\phi$, $\sqrt{x^2+y^2+(z+a)^2} = 2a \cos \phi/2$.

So, $$\iint_S \frac{d\sigma}{\sqrt{x^2+y^2+(z+a)^2}} = \int_0^{2\pi}\int_0^{\pi/2} \frac{a^2 \sin\phi}{2a \cos \phi/2}\, d\phi\, d\theta \ //.$$

(2) Evaluate the surface integral $\iint_S g\, d\sigma$ where $g(x,y,z) = x+y+z$ and the surface S is described by $z = 2x+3y$, $x \ge 0$, $y \ge 0$, $x+y \le 2$.

The projection T of the surface is $\{(x,y) \mid x \ge 0, y \ge 0, x+y \le 2\}$.

$$\iint_S g\, d\sigma = \iint_T (x+y+z)\sqrt{1+f_x^2+f_y^2}\, dx\, dy$$

$$= \int_0^2 \int_0^{2-y} (x+y+2x+3y)\sqrt{14}\, dx\, dy \ //.$$