MA1012: Problem Sheet 2

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1. Find the following limits (if they exist):
   (a) $\lim\limits_{(x,y)\to(0,0)} \dfrac{3x^2|y|}{x^2+y^2}$,
   (b) $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2}$,
   (c) $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{(x+y+z)^2}{x^2+y^2+z^2}$.
2. Using the sequential definition of limits, show that the function $f:\mathbb{R}^2\to\mathbb{R}$ defined by
$$f(x,y)=\begin{cases}\dfrac{\sin(xy)}{xy} & xy\neq 0,\\ 1 & xy=0,\end{cases}$$
   is continuous on $\mathbb{R}^2$.
3. Let $f:\mathbb{R}^3\to\mathbb{R}$ be given by
$$f(x,y)=\begin{cases}1 & xy=0,\\ 0 & xy\neq 0.\end{cases}$$
   Show that $f_x(0,0)=f_y(0,0)=0$. Is f continuous at $(0,0)$?
4. Consider the function $f:\mathbb{R}^2\to\mathbb{R}$ given by
$$f(x,y)=\begin{cases}\dfrac{3x^2y-y^3}{x^2+y^2} & (x,y)\neq(0,0)\\ 0 & (x,y)=(0,0).\end{cases}$$
   Is f continuous at $(0,0)$? Is f_y continuous at $(0,0)$?
5. Are the following functions differentiable at $(0,0)$ where $f:\mathbb{R}^2\to\mathbb{R}$ and $f(x,y)$ is given by
   (a) $|x|+|y|$,
   (b) $\sqrt{|xy|}$,
   (c) $|xy|$,
   (d) $x^2+\sin y+y^2e^x$.
6. Let $f:\mathbb{R}^2\to\mathbb{R}$ be differentiable, and suppose $f(1,2)=f(3,4)=0$, then show that there exists a point (x_0,y_0) lying in the line segment joining $(1,1)$ and $(3,4)$ such that $f_x(x_0,y_0)+f_y(x_0,y_0)=0$.