# MA1012: Problem Sheet 3

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March 2023

1. Determine in which directions the directional derivatives exists for the following functions $f : \mathbb{R}^2 \to \mathbb{R}$, where $f(x, y)$ is given by
   (a) $\sqrt{|xy|}$, and
   (b) $\dfrac{x^2y}{x^2+y^2}$.
2. For $X \in \mathbb{R}^3$ and $f(X) := ||X||$, let $X_0 = (x_0, y_0, z_0) \in \mathbb{R}^3$ and $||X_0|| = 1$, show that $\nabla f(X_0) = X_0$ and find the equation of the tangent plane of the sphere $f(x, y, z) = 1$ at $X_0$.
3. Find a point on the surface $z = g(x, y) = x^2 - 2xy + 2y$ at which the surface has a horizontal tangent plane.
4. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be differentiable and $M \in \mathbb{R}$ be such that $|f_x(X)| \leq M$ and $|f_y(X)| \leq M$ for all $X \in \mathbb{R}^2$, then show that for all $X, Y \in \mathbb{R}^2$ we have $|f(X) - f(Y)| \leq 2M||X - Y||$.
5. Prove the extended mean value theorem.
6. Let $D \subset \mathbb{R}^2$ and $(x_0, y_0)$ be an interior point of $D$. Suppose that $f : D \to \mathbb{R}$ and $f$ has a local maximum or minimum at $(x_0, y_0)$. Then,
   (a) if $(u, v) \in \mathbb{R}^2$ with $||(u, v)|| = 1$ and $D_{(x_0,y_0)}f(u, v)$ exists then $D_{(x_0,y_0)}f(u, v) = 0$, and
   (b) if $f$ is differentiable at $(x_0, y_0)$, then $f'(x_0, y_0) = 0$.
7. Examine the behaviour of the following functions at the point $(0, 0)$:
   (a) $x^2 - y^2$, and
   (b) $x^2 - 2xy^2$.
8. Find a point on the surface $z = xy + 1$ that is nearest to the origin.
9. Find the points of global maximum and global minimum of the function $f(x, y) = x^2 + y^2 - 2x + 2$ on the region $\{(x, y) : x^2 + y^2 \leq 4, y \geq 0\}$.
10. Prove the second derivative test.