~~Curl~~, Divergence and Stoke's Theorem

Let $F: \mathbb{R}^3 \to \mathbb{R}^3$, $F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$.

Such fns are called vector fields.

Def^n: The curl of F is a vector field denoted by curl F and defined by

$$\text{curl} F = \left(R_y - Q_z,\ P_z - R_x,\ Q_x - P_y\right)$$

$$= \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times f.$$

$$\downarrow$$

$$\nabla = \frac{\partial}{\partial x} i + \frac{\partial}{\partial y} j + \frac{\partial}{\partial z} k$$

(gradient field)

Def^n: The divergence of F is a scalar valued fn denoted by div F and is ~~denoted~~ defined by

$$\text{div} F = P_x + Q_y + R_z = \nabla \cdot F.$$

Def^n: Let $S = r(u,v)$ be a parametric surface defined on a parameter domain T. We say that S is smooth if r_u and r_v are cont. on T and $r_u \times r_v \neq 0$ on T. A level surface S defined by $f(x,y,z) = c$ is said to be smooth if ∇f is cont and never zero on S.

Def^n: A smooth surface is said to be orientable if there exists a cont. unit normal vector fn. defined at each pt. of the surface. eg: spheres, planes, etc.

Non-orientable: Möbius strip.

Orientation on the boundary w.r.t. a normal: Let $S = r(u,v)$ be a parametric orientable surface defined on the parameter domain T. Consider the unit normal $n = \frac{r_u \times r_v}{\|r_u \times r_v\|}$ of the surface S. Let Γ be the boundary of T and $C = r(\Gamma)$. If we assume Γ is oriented in the counterclockwise dir^{n} then we get an orientation for C inherited from Γ through the mapping r. This orientation for C is considered to be the orientation w.r.t n.

Recall, $\iint_D (N_x - M_y)\,dx\,dy = \oint_C M\,dx + N\,dy$, where D is a plane enclosed by a simple closed curve C.

Consider $F: \mathbb{R}^2 \to \mathbb{R}^2$, $F(x,y) = (M(x,y), N(x,y))$.

$\text{curl } F = \nabla \times F = (N_x - M_y)k$. So, Green's theorem is

$$\iint_D (\text{curl } F).k\,dx\,dy = \oint_C F.dR.$$

We now extend this.

Stokes' Theorem: Let S be a piecewise smooth orientable surface and C be its boundary, which is a piecewise closed curve. Let $F(x,y,z) = (P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field s.t. P, Q and R are cont. and have cont. first partial derivatives in an open set containing S. If n is a unit normal to S, then we have

$$\iint_S (\text{curl } F).n\,d\sigma = \oint_C F.dR,$$

where the line integral is taken around C in the dir^{n} of the orientation of C w.r.t. n.

• The value of the line int. depends only on C, so the shape of S is immaterial.

• If S is a plane region then Stoke's theorem becomes Green's theorem.

• For a closed oriented surface such as a sphere, there is no boundary, so, $\iint_S (\text{curl } F)\cdot n\, d\sigma = 0$.

• We can extend this to a smooth surface which has more than one simple closed curve forming the boundary of the surface.

Q: Let S be the part of the cylinder $z = 1-x^2$, $0 \le x \le 1$, $-2 \le y \le 2$. Let C be the boundary curve of S. Let $F(x,y,z) = (y, y, z)$. Find the unit outer normal to S, curl F and evaluate $\oint_C F\cdot dR$, where C is oriented clockwise as viewed from above the surface.

Sol^n: The surface is, $z = 1-x^2$. We consider this as the graph of $f(x,y) = 1-x^2$, or a parametric surface $r(x,y) = (x, y, f(x,y))$.

An unit normal is $\dfrac{r_x \times r_y}{\| r_x \times r_y \|} = \dfrac{-f_x - f_y + k}{\sqrt{1+f_x^2+f_y^2}} = \dfrac{2xi+k}{\sqrt{1+4x^2}}$.

$\text{curl } f = -k$; $\text{curl } f \cdot n = \dfrac{-1}{\sqrt{1+4x^2}}$.

By Stoke's Theorem we have,

$$\oint_C F\cdot dR = \iint_S \frac{-1}{\sqrt{1+4x^2}}\, d\sigma = \iint_R \frac{-1}{\sqrt{1+4x^2}} \sqrt{1+f_x^2+f_y^2}\, dx\, dy$$

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Let D be a plane region enclosed by a simple smooth curve C. Let $F(x,y) = (M(x,y), N(x,y))$ s.t. M and N satisfy the conditions of Green's Theorem.

If the curve C is defined by $R(t) = (x(t), y(t))$, then the vector $n = \left(\frac{dy}{ds}, \frac{dx}{ds}\right)$ is a unit normal to the curve C because, $T = \left(\frac{dx}{ds}, \frac{dy}{ds}\right)$ is a unit tangent to the curve C. By Green's theorem we have,

$$\oint_C (F.n)\,ds = \oint_C M dy - N dx = \iint_D (M_x - (-N_y))\,dx\,dy$$

$$= \iint_D (M_x + N_y)\,dx\,dy.$$

This gives us, $\oint_C (F.n)\,ds = \iint_D \operatorname{div} F\,dx\,dy$.

<u>Divergence Theorem:</u> Let D be a solid in $\mathbb{R}^3$ bdd. by a piecewise smooth orientable surface S. Let $F(x,y,z)$ be $(P(x,y,z), Q(x,y,z), R(x,y,z))$ be a vector field s.t. P, Q and R are cont. and have cont. first partial derivatives in an open set containing D. Suppose n is the unit outward normal to S. Then we have,

$$\iiint_D \operatorname{div} F.\,dV = \iint_S F.n\,d\sigma.$$

<u>Q.</u> Evaluate the surface integral $\iint_S F.n\,d\sigma$ where $F(x,y,z) = (x+y, z^2, x^2)$ and S is the surface of the hemisphere $x^2+y^2+z^2 = 1$, with $z > 0$ and n is the outward normal to S.

Sln: The surface is not closed.

$D := x^2+y^2+z^2 \le 1,\ z \geq 0,$

$$\iiint_D \operatorname{div} F . dV = \iint_S F.n\, d\sigma + \iint_{S_1} F.n_1\, d\sigma,$$

where $S_1 := x^2+y^2<1,\ z=0$ is the base of the hemisphere

n_1 is the outward unit normal to $S_1 = -k$.

$\operatorname{div} F = 1$, so the volm ~~of the~~ integral in the above eqn is the volm of the hemisphere $= 2\pi/3$.

So, $$2\pi/3 = \iint_S F.n\, d\sigma + \iint_{S_1} F.-k\, d\sigma$$

$$\Rightarrow \iint_S F.n\, d\sigma = 2\pi/3 - \iint_{S_1} F.-k\, d\sigma = 2\pi/3 + \iint_{S_1} x^2\, d\sigma.$$

Let $S_1 = (r\cos\theta,\ r\sin\theta)$, $0 \le r < 1$, $0 \le \theta \le 2\pi$.

$$\iint_{S_1} x^2\, d\sigma = \int_0^1\int_0^{2\pi} r^2\cos^2\theta\, r\, d\theta\, dr = \int_0^1 r^3\pi\, dr = \pi/4.$$

So, $$\iint_S F.n\, d\sigma = 2\pi/3 + \pi/4 = 11\pi/12.$$ //