Theorems on functions of Several Variables

Mixed Derivative Theorem: If $f(x,y)$ and ~~its~~ f_x, f_y, f_{xy}, f_{yx} are defined in a neighbourhood of (x_0, y_0) and all are continuous at (x_0, y_0) then $f_{xy}(x_0, y_0) = f_{yx}(x_0, y_0)$.
(Proof is an application of MVT - Consult the book.)

Mean Value Theorem: Suppose $f: \mathbb{R}^2 \to \mathbb{R}$ is diff. Let $X_0 = (x_0, y_0)$ and $X = (x_0+h, y_0+k)$. Then there exists C which lies on the line joining X_0 and X s.t. $f(X) = f(X_0) + f'(C)\,(X - X_0)$.
That is, $\exists\, c \in (0,1)$ such that

$$f(x_0+h, y_0+k) = f(x_0, y_0) + h f_x(C) + k f_y(C) \text{ where}$$
$$C = (x_0 + ch, y_0 + ck).$$

Proof: We define $\varphi: [0,1] \to \mathbb{R}$ by,

$$\varphi(t) = f(x_0 + th, y_0 + tk), \quad t \in [0,1].$$

By chain rule, $\varphi'(t) = f_x \dfrac{dx}{dt} + f_y \dfrac{dy}{dt} = h f_x + k f_y$.

By the MVT, $\exists\, c \in (0,1)$ s.t. $\varphi(1) - \varphi(0) = \varphi'(c)\,(1-0)$.

This proves the result. //

Extended Mean Value Theorem: Let f, X, X_0 be the same as before. Suppose f_x and f_y are continuous and they have continuous partial derivatives. Then, $\exists\, C$ which lies on the line joining X_0 and X such that

$$f(X) = f(X_0) + f'(X_0)(X - X_0) + \frac{1}{2}(X - X_0) f''(C)(X - X_0).$$

Here $f'' = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$.

That is, $\exists\, \bar{t} \in (0,1)$ such that

$$f(x_0+h, y_0+k) = f(x_0, y_0) + (hf_x + kf_y)(X_0) + \frac{1}{2}(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy})\, C$$

with $C = (x_0 + \bar{t}h, y_0 + \bar{t}k)$.

<u>Proof:</u> Take φ to be the same fn as before.
Since f_x & f_y are continuous so f is diff. So, φ is diff and we have, $\varphi' = hf_x + kf_y$.
Since f_x & f_y have continuous partial derivatives, they are diff. We denote

$$\varphi'(t) = hf_x(x_0 + th, y_0 + tk) + k\, f_y(x_0 + th, y_0 + tk)$$
$$= F(x_0 + th, y_0 + tk), \quad t \in [0,1].$$

We apply the chain rule to get,

$$\varphi'' = hF_x + kF_y = h\frac{\partial(hf_x + kf_y)}{\partial x} + k\frac{\partial}{\partial y}(hf_x + kf_y)$$
$$= h\left(k\frac{\partial^2 f}{\partial y \partial x} + h\frac{\partial^2 f}{\partial x^2}\right) + k\left(h\frac{\partial^2 f}{\partial x \partial y} + k\frac{\partial^2 f}{\partial y^2}\right)$$
$$= h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy} \quad \text{(By the Mixed Derivative Thm).}$$

By the extended MVT for φ, $\exists\, \bar{t} \in (0,1)$ s.t.

$$\varphi(1) = \varphi(0) + \varphi'(0) + \frac{\varphi'(\bar{t})}{2}.$$

We now replace $\varphi, \varphi', \varphi''$ in the above to get the result //.

- f'' is called the <u>Hessian matrix</u>. For a fn of n variables, the Hessian matrix is $H_f(x) = \left(\dfrac{\partial^2 f}{\partial x_i \partial x_j}\right)_{n\times n}$ $x = (x_1, x_2, \ldots, x_n)$

Multivariable Version of Taylor's Theorem: Let $f: \mathbb{R}^n \to \mathbb{R}$ be a k times continuously differentiable fn. at the point $a = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n$. Then there exists functions $h_\alpha : \mathbb{R}^n \to \mathbb{R}$, with $|\alpha| = k$, such that

$$f(x) = \sum_{|\alpha| \le k} \frac{D^\alpha f(a)}{\alpha!} (x-a)^\alpha + \sum_{|\alpha| = k+1} h_\alpha(x)(x-a)^\alpha,$$

and $\lim_{x \to a} h_\alpha(x) = 0$, here the following notations are used:

$\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$, $\alpha! = \alpha_1! \alpha_2! \cdots \alpha_n!$,

$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$, $D^\alpha f = \dfrac{\partial^{|\alpha|} f}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$, $|\alpha| \le k$.

Maxima & Minima:

- A non-empty subset $D \subseteq \mathbb{R}^n$ is said to be closed if a seq^n in D converges then its limit point lies in D.

eg: $D_1 = \{ X_0 \in \mathbb{R}^2 : \|X\| \le 1 \}$, $D_2 = \{ (x,y) \in \mathbb{R}^2 : x \geq 0, y \geq 0 \}$ closed.

$D_3 = \{ X \in \mathbb{R}^2 : \|X\| < 1 \}$, $D_4 = \{ (x,y) \in \mathbb{R}^2 : x > 0, y > 0 \}$ NOT closed.

Def^n: $D \subseteq \mathbb{R}^n$, $X_0 \in D$, we say that X_0 is an interior pt. of D if $\exists\, r > 0$ s.t. the neighbourhood $N_r(X_0) = \{ X \in \mathbb{R}^n : \|X_0 - X\| < r \}$ is contained in D.

eg: all pts of D_3 are interior pts of D_1.

<u>Theorem</u>: Let D be a closed and bdd. subset of $\mathbb{R}^2$ and $f: D \to \mathbb{R}$ be cont. Then f has a max$^{\underline{m}}$ and a min$^{\underline{m}}$ in D.

<u>Theorem: (Necessary Cond$^{\underline{n}}$)</u>: $D \subseteq \mathbb{R}^2$, $f: D \to \mathbb{R}$, (x_0, y_0) is an interior pt. of D. Let f_x and f_y exist at (x_0, y_0). If f has a local max$^{\underline{m}}$ or local min$^{\underline{m}}$ at (x_0, y_0) then $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

<u>Proof</u>: The single variable fns $f(x, y_0)$ and $f(x_0, y)$ have local max$^{\underline{m}}$ or min$^{\underline{m}}$ at x_0 and y_0 resp. So, the derivatives of these fns are 0 at x_0 and y_0 resp. That is $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. //

- The above is NOT sufficient.

<u>eg</u>: $f: \mathbb{R}^2 \to \mathbb{R}$, $f(x,y) = xy$, $f_x(0,0) = f_y(0,0) = 0$ but $(0,0)$ is neither a local max$^{\underline{m}}$ nor a local min$^{\underline{m}}$.

<u>Second Derivative Test</u>: $D \subseteq \mathbb{R}^2$ and $f: D \to \mathbb{R}$. Suppose f_x, f_y are cont. and they have cont. partial derivatives on D. Let (x_0, y_0) be an interior pt. of D and $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Further suppose $(f_{xx} f_{yy} - f_{xy}^2)(x_0, y_0) > 0$. Then,

(i) if $f_{xx}(x_0, y_0) > 0$ then f has a local min$^{\underline{m}}$ at (x_0, y_0)

(ii) if $f_{xx}(x_0, y_0) < 0$ then f has a local max$^{\underline{m}}$ at (x_0, y_0).

<u>Proof of (i)</u>: Suppose $f_{xx}(x_0, y_0) > 0$ and $(f_{xx} f_{yy} - f_{xy}^2)(x_0, y_0) > 0$. Then $\exists$ a neighbourhood of (x_0, y_0), say N such that

$$f_{xx}(x,y) > 0, \ (f_{xx} f_{yy} - f_{xy}^2)(x,y) > 0 \quad \forall (x,y) \in N.$$

Let $(x_0+h, y_0+k) \in N$, by the Extended MVT, there exists some C lying in the line joining (x_0+h, y_0+k) and (x_0, y_0) such that

$$f(x_0+h, y_0+k) - f(x_0, y_0) = \frac{1}{2}\left(h^2 f_{xx} + 2hk f_{xy} + k^2 f_{yy}\right)C$$
$$:= Q(C).$$

we have,

$$\left[(hf_{xx} + kf_{xy})(C)\right]^2 + k^2(f_{xx}f_{yy} - f_{xy}^2)(C)$$
$$= 2f_{xx}(C)Q(C) > 0$$

Since $f_{xx}(C) > 0$ we have $Q(C) > 0$ and so,

$f(x_0+h, y_0+k) > f(x_0, y_0)$ so f has a local min^m at (x_0, y_0). //.

- We cannot apply this test when $f_x(x_0, y_0) = f_y(x_0, y_0)$ $= (f_{xx}f_{yy} - f_{xy}^2)(x_0, y_0) = 0$

eg: $\left.\begin{array}{l} f_1(x,y) = -x^4 - y^4 \\ f_2(x,y) = x^4 + y^4 \end{array}\right\}$ both satisfy this at $(0,0)$.

f_1 has local max^m at $(0,0)$

f_2 has local min^m at $(0,0)$.

<u>Saddle Point:</u>

- An interior point of the domain of a fn $f(x,y)$ where both f_x, f_y are zero or where one or both f_x, f_y do not exist is a <u>critical pt.</u> of f.

- A diff. fn $f(x,y)$ has a <u>saddle point</u> at a critical pt. (a,b) if in every open set centered at (a,b) there are domain pts (x,y) where $f(x,y) > f(a,b)$ and domain pts (x,y) where $f(x,y) < f(a,b)$. The pt. $(a,b,f(a,b))$ on the surface $z = f(x,y)$ is called a saddle pt. of the surface.

Remark: If (x_0, y_0) is an interior pt. of D, and we have $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ and $(f_{xx}f_{yy} - f_{xy}^2)(x_0, y_0) < 0$, then (x_0, y_0) is a saddle point.

eg: $f(x, y) = x\sin y$, $(x_0, y_0) = (0, n\pi)$.

Problems:

1. Find the local extreme values of $f(x, y) = x^2 + y^2$.

Sol^n: $f_x = 2x$, $f_y = 2y$, they exist everywhere.
So, local extreme value occurs only when $f_x = f_y = 0$
$\Rightarrow (x, y) = (0, 0)$.

Since $f \geq 0$, so $(0,0)$ is a local min^m. //.

2. Let $f(x, y) = 3x^4 - 4x^2y + y^2$. Does f have a local min^m at $(0,0)$ along the line through $(0,0)$? Does f have a min^m at $(0,0)$? Is $(0,0)$ a saddle pt. of f?

Sol^n: Along the x-axis, the local min^m of the fn is at $(0,0)$.
Let $x = r\cos\theta$, $y = r\sin\theta$, for $\theta \neq 0, \pi$, (or let $y = mx$).
$f(r\cos\theta, r\sin\theta)$ is a fn of one variable. By second derivative test we see that $(0,0)$ is a local min^m.

$f(x, y) = (3x^2 - y)(x^2 - y)$, in the region bet^n the parabolas $3x^2 = y$, $y = x^2$, the fn takes negative values and is positive otherwise. So, $(0,0)$ is a saddle pt. //.

3. Find the local extreme value of $f(x, y) = xy$.

Sol^n: $f_x = x$, $f_y = 0$. $f_{xx} = 0$, $f_{yy} = 0$, $f_{xy} = 1$.
$f_{xx}f_{yy} - f_{xx}^2 = -1 < 0$.
So, $(0,0)$ is a saddle pt. //.