Triple Integrals: The def^n is analogous to that of double integrals, now defined over $Q = [a,b] \times [c,d] \times [e,f]$.

If f on Q is integrable then the integral called the triple integral is denoted by $\iiint_Q f(x,y,z)\,dx\,dy\,dz$ or $\iiint_Q f(x,y,z)\,dV$.

- Every cont. fn on Q is integrable.
- We can extend this def^n to any bdd region in $\mathbb{R}^3$.
- There is no geometric interpretation here like the vol^m interpretation for double integrals.
- $\iiint_D dx\,dy\,dz$ is taken as the vol^m of the region D.

Fubini's Theorem: Let D be a bdd domain in $\mathbb{R}^3$,

$$D = \{(x,y,z) \mid (x,y) \in R \text{ and } f_1(x,y) \le z \le f_2(x,y)\}.$$

Thus, D is bdd above by the surface $z = f_1(x,y)$ and bdd below by the surface $z = f_2(x,y)$, and on the side by the cylinder generated by a line moving $\parallel$ to z-axis along the boundary of R.

The projection of D on the xy-plane is the region R.

If f is cont. on D and f_1, f_2 are cont. on R, then we have

$$\iiint_D f(x,y,z)\,dV = \iint_R \left(\int_{f_1(x,y)}^{f_2(x,y)} f(x,y,z)\,dz \right) dA.$$

eg: D is the region in space bounded by $x=0$, $y=0$, $z=2$ and the surface $z = x^2+y^2$.

$$D = \{(x,y,z) \mid (x,y) \in R,\ x^2+y^2 \le z \le 2\},$$

$$R = \{(x,y) \mid 0 \le x \le \sqrt{2},\ 0 \le y \le \sqrt{2-x^2}\}.$$

$$\iiint_D x\,dx\,dy\,dz = \iint_R \left(\int_{x^2+y^2}^{2} x\,dz \right) dA = \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} x\,dx\,dy\,dz.$$

$$= 8\sqrt{2}/15 \,.\, //.$$

• Triple integrals satisfy the same algebraic properties as double and single integrals.

Change of variables:

$$\iiint_S f(x,y,z)\,dx\,dy\,dz = \iiint_T f(X(u,v,w), Y(u,v,w), Z(u,v,w)) \times |J(u,v,w)|\,du\,dv\,dw,$$

where $J(u,v,w) = \begin{vmatrix} \partial X/\partial u & \partial Y/\partial u & \partial Z/\partial u \\ \partial X/\partial v & \partial Y/\partial v & \partial Z/\partial v \\ \partial X/\partial w & \partial Y/\partial w & \partial Z/\partial w \end{vmatrix}$.

Cylindrical coordinates: $x = X(r,\theta) = r\cos\theta$

$y = Y(r,\theta) = r\sin\theta$

$z = z$.

Assume $r > 0$, $\theta \in [0, 2\pi)$.

$J(u,v,z) = r. \Rightarrow \iiint_S f(x,y,z)\,dV = \iiint_T f(r\cos\theta, r\sin\theta, z) \times r\,dr\,d\theta\,dz.$

eg: Evaluate $\iiint_D (z^2x^2 + z^2y^2)\,dx\,dy\,dz$ where D is the region determined by $x^2+y^2 \le 1$, $-1 \le z \le 1$.

- In cylindrical coordinates, $0 \le r \le 1$, $0 \le \theta \le 2\pi$, $-1 \le z \le 1$,

so, the integral is $\int_{-1}^{1} \int_0^{2\pi} \int_0^{1} (z^2 r^2)\, r\,dr\,d\theta\,dz = \int_{-1}^{1} \int_0^{2\pi} z^2 \left[\frac{r^4}{4}\right]_0^1 d\theta\,dz$

$$= \int_{-1}^{1} \frac{2\pi}{4} z^2\,dz = \pi/3 \,.\, //.$$

Spherical Coordinates: Given a pt. (x, y, z), let $\rho = \sqrt{x^2+y^2+z^2}$, ϕ is the angle that (x, y, z) makes with the z-axis.

$$z = \rho\cos\phi, \quad x = \rho\sin\phi\cos\theta, \quad y = \rho\sin\phi\sin\theta,$$

where $x = r\cos\theta$, $y = r\sin\theta$ are the expressions in polar coordinates.

Assume $\rho > 0$, $0 \le \theta \le 2\pi$, $0 \le \phi < \pi$. (Note $r = \rho\sin\phi$)

$$J(\rho, \theta, \phi) = -\rho^2\sin\phi.$$

$$\iiint_S f(x,y,z)\,dV = \iiint_T f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\,\rho^2\sin\phi\, d\rho\, d\theta\, d\phi.$$

eg: Evaluate $\iiint_D \frac{z}{(x^2+y^2+z^2)^{3/2}}\,dV$, where $D = \{(x,y,z) \mid x^2+y^2+z^2 \le 4a^2,\ z \ge a\}$.

- In spherical coordinates, ϕ varies from 0 to $\pi/3$, $0 \le \theta \le 2\pi$, $a\sec\phi \le \rho \le 2a$. So the reqd. integral is,

$$\int_0^{\pi/3}\int_0^{2\pi}\int_{a\sec\phi}^{2a} \frac{\cos\phi}{\rho^2}\,|J(\rho,\theta,\phi)|\, d\rho\, d\theta\, d\phi$$

$$= 2\pi\int_0^{\pi/3} (2a\sin\phi\operatorname{cosec}\phi - a\sin\phi)\,d\phi = \pi a/2.$$