~~Review of vectors, eqns of lines~~

## Vector valued functions:

- Each vector valued fn $F$ is associated with three real valued functions $f_1, f_2, f_3$ and we write $F = (f_1, f_2, f_3)$.

eg: $F_1(t) = (\cos t, \sin t)$, $0 \le t \le 2\pi$ ~ varies on a circle

$F_2(t) = (\cos t, \sin t, t)$, $\infty < t < \infty$. ~ varies on a helix.

Parametric curves: $I \subset \mathbb{R}$, an interval and $F : I \to \mathbb{R}^3$.

The set of points $\{F(t) : t \in I\}$ is called the graph of $F$. If $F$ is continuous then such a graph is called a curve or parametric curve with parameter $t$.

- Each cont. vector valued fn corresponds to a curve.
- eg Let $X_0, P \in \mathbb{R}^3$, $P \neq 0$. $F(t) = X_0 + tP$.

range = line through $X_0$ $\parallel$ to $P$.

- Let $F = (f_1, f_2, f_3)$ be a vector valued fn and $L = (l_1, l_2, l_3)$.
- $\lim\limits_{t \to t_0} F(t) = L$ if $\lim\limits_{t \to t_0} \|F(t) - L\| = 0$.

Proposition: $\lim\limits_{t \to t_0} F(t) = L$ iff $\lim\limits_{t \to t_0} f_i(t) = l_i$ for $i = 1, 2, 3$.

Proof: $\sum\limits_{i=1}^{3} |f_i(t) - l_i| \to 0$ iff $|f_i(t) - l_i| \to 0$, $i = 1, 2, 3$.

- We say that $F$ is continuous at $t_0$ if $\lim\limits_{t \to t_0} F(t) = F(t_0)$.
- $F$ is continuous at $t_0$ iff each of the component fns $f_i$ is continuous at $t_0$.
- $F$ is differentiable at $t_0$ if $\lim\limits_{h \to 0} \dfrac{F(t_0 + h) - F(t_0)}{h}$ exists.

The limit is then $F'(t_0) = (f_1'(t_0), f_2'(t_0), f_3'(t_0))$.

<u>Tangent Vector</u>: Suppose $F$ is diff at $t_0$ and $F'(t_0) \neq 0$.

Then, $F'(t_0) = \lim_{h\to 0} \underbrace{\frac{1}{h}(F(t_0+h) - F(t_0))}_{}$.

this vector is parallel to $F(t_0+h) - F(t_0)$

moves to a tangent vector as $h \to 0$.

Suppose $C$ is a curve defined by a diff. vector valued fn $R$. Let $R'(t_0) \neq 0$, then the vector $R'(t_0)$ is called a <u>tangent vector to $C$ at $R(t_0)$</u> and the line $X(t) = R(t_0) + tR'(t_0)$ is called the <u>tangent line to $C$ at $R(t_0)$</u>.

<u>Arc lengths</u>: Let $C$ be a space curve defined by $R(t) = (x(t), y(t), z(t))$, $a \le t \le b$. The length of $C$ is defined as
$$L = \int_a^b \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}\, dt = \int_a^b \left\| \frac{dR}{dt} \right\|.$$

- Let $R(t_0)$ be a fixed pt. on $C$, for $t$, the directed distance measured along $C$ from $R(t_0)$ and upto $R(t)$ is
$$s(t) = \int_{t_0}^{t} \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}\, d\tau.$$

- Each value of $s$ corresponds to a fixed pt. on $C$ and this parametrizes $C$ w.r.t. $s$, the <u>arc length parameter</u>.

Clearly, $\frac{ds}{dt} = \left\| \frac{dR}{dt} \right\|$.

<u>Unit tangent vector</u>: of $R(t)$ is $T = \frac{R'(t)}{\|R'(t)\|}$, $\|R'(t)\| \neq 0$.

$$\Rightarrow T = \frac{dR/dt}{ds/dt} = \frac{dR}{dt} \cdot \frac{dt}{ds} = \frac{dR}{ds}.$$

**Theorem:** Let $f : [a,b] \to \mathbb{R}$, $f'(x) \neq 0 \ \forall x \in [a,b]$, then $f^{-1}$ is cont., diff$^n$ and $(f^{-1})'(f(x)) = \dfrac{1}{f'(x)}$. (Recall).

So we have, $\dfrac{dt}{ds} = \dfrac{1}{\frac{ds}{dt}}$.

**Theorem:** Let $I \subset \mathbb{R}$, an interval, $F$ is vector valued on $I$ s.t. $\|F(t)\| = \alpha \ \forall t \in I$. Then $F \cdot F' = 0$ on $I$, i.e. $F'(t)$ . $\perp^r$ to $F(t)$.

**Proof:** Let $g(t) = \|F(t)\|^2 = F(t) \cdot F(t)$.

$g$ is constant on $I$ so, $g' = 0$ on $I$.

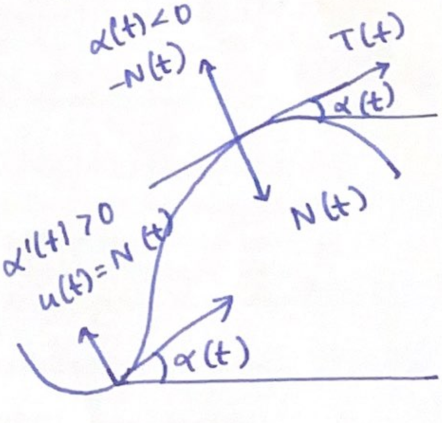
$g' = F \cdot F' + F' \cdot F = 2F \cdot F' \Rightarrow F \cdot F' = 0$ //.

• The unit vector $T$ has length 1, so $T \cdot T' = 0$ by the th$^m$.

We define the principle normal to the curve $N(t) = \dfrac{T'(t)}{\|T'(t)\|}$, whenever $\|T'(t)\| \neq 0$.

Consider a plane curve with $T$ as a unit vector.

Say $T(t) = (\cos\alpha(t), \sin\alpha(t))$, $\alpha(t)$ is the angle bet$^n$ the tangent vector and $x$-axis.




we have,

$$T'(t) = (-\sin\alpha(t)\alpha'(t), \cos\alpha(t)\alpha'(t))$$
$$= \alpha'(t)\, u(t),$$

where $u(t) = [\cos(\alpha(t) + \pi/2), \sin(\alpha(t) + \pi/2)]$, another unit vector.

When $\alpha'(t) > 0$, the angle is increasing and $N(t) = u(t)$.

When $\alpha'(t) < 0$, - - - - - decreasing and $N(t) = -u(t)$.

Curvature of a curve: (Related to rate of change of the unit tangent w.r.t. the arc length).

$$K = \left\|\frac{dT}{ds}\right\|.$$

We have, $\dfrac{dT}{ds} = \dfrac{dT}{dt}\cdot\dfrac{dt}{ds} = \dfrac{T'(t)}{\left\|\frac{dR}{ds}\right\|} \Rightarrow k(t) = \dfrac{\|T'(t)\|}{\left\|\frac{dR}{ds}\right\|}.$

eg: Let $T(t) = (\cos\alpha(t), \sin\alpha(t))$.

$$\frac{d\alpha}{dt} = \frac{d\alpha}{ds}\cdot\frac{ds}{dt} = \left\|\frac{dR}{dt}\right\|\frac{d\alpha}{ds}.$$

So, $k(t) = \left|\dfrac{d\alpha}{ds}\right|$.

- Let $R(t) = (a\cos t, a\sin t)$, $R'(t) = (-a\sin t, a\cos t)$

  $T(t) = (-\sin t, \cos t)$, $T'(t) = (-\cos t, -\sin t)$.

  $\|R'(t)\| = a$, $\|T'(t)\| = 1$, So, $k = 1/a$.

  (The circle has constant curvature.)

Theorem: Let $v(t)$ and $a(t)$ denote the velocity and the acceleration vectors of a motion of a particle on a curve defined by $R(t)$. Then, $k(t) = \dfrac{\|a(t)\times v(t)\|}{\|v(t)\|^3}$.

- The curvature of a plane curve is defined to ~~too~~ be the rate of change of the angle betn the tangent vector and the positive $x$-axis.

Problem: A plane curve has the Cartesian eq$^n$ $y = f(x)$, where $f$ is twice diff. What is the curvature at $(x, f(x))$?.

Sol$^n$: Consider the graph $R(t) = (t, f(t))$.

$$v(t) = R'(t) = (1, f'(t))$$
$$a(t) = R''(t) = (0, f''(t)).$$

So, $\|v(t) \times a(t)\| = |f''(t)|$

$$\|v(t)\| = \sqrt{1 + f'(t)^2} \;. \;//$$

Problem: Let $R(t) = (t, t^2, \frac{2}{3}t^3)$. Find the tangent and principal normal at $t = 1$. What is the curvature of the curve?

Sol$^n$: $R'(t) = (1, 2t, 2t^2)$, $\quad T(t) = \dfrac{R'(t)}{\|R'(t)\|} = \left(\dfrac{1}{1+2t^2}, \dfrac{2t}{1+2t^2}, \dfrac{2t^2}{1+2t^2}\right).$

$$T'(t) = \left(\frac{-4t}{(1+2t^2)^2}, \frac{(2-4t^2)}{(1+2t^2)^2}, \frac{4t}{(1+2t^2)^2}\right)$$ is the dir$^n$ of the normal.

At $t = 1$, unit tangent is $T(1) = \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3}\right)$.

normal vector is $T'(1) = \left(\frac{-4}{9}, \frac{-2}{9}, \frac{4}{9}\right)$.

So the eq$^n$ of the tangent is $(x, y, z) = (1, 1, 2/3) + t(1, 2, 2)$

The ~~dir~~ eq$^n$ of the normal is $(x, y, z) = (1, 1, 2/3) + t(-2, -1, 2)$.

$$k(t) = \frac{\|T'(t)\|}{\left\|\frac{dR}{ds}\right\|} = \frac{2}{(1+2t^2)^2} \;. \;//.$$