Indian Institute of Information Technology (IIIT) Manipur
End Semester Examination, May 2023

Course Title: **Optimization Techniques** Course Code: **MA301**

Semester: CSE VI Maximum Marks: 100

Date of Examination: 1 May 2023 Time: 180 minutes

Write legibly and show your full work to get credit.

Part A (10×2 marks = 20 marks)

Instructions

- All questions are compulsory in this part.

Questions

1. Using a numerical example verify that the function $f(x) = \sqrt{x}$ with $x \geq 0$ is not convex. (**2 marks**)
2. What is alternate minima in the context of an LPP? (**2 marks**)
3. Construct the dual of the following problem

$$\text{Max} \quad c^T x$$

subject to

$$\begin{aligned} Ax &\leq b, \\ (-A)x &\leq -b, \\ x &\geq 0. \end{aligned}$$

(**2 marks**)

4. Prove that if the primal and dual both have feasible solutions then both have optimal solutions. (**2 marks**)
5. In the context of a balanced transportation problem, prove that a loop has an even number of cells. (**2 marks**)
6. In the context of the Hungarian algorithm, define independent zeroes. (**2 marks**)
7. In the context of a convex set S and a function $f : S \to \mathbb{R}$, define what is the epigraph and draw a diagram to illustrate the definition. (**1+1 marks**)
8. Let $S \subset \mathbb{R}^n$ be a convex set and $f : S \to \mathbb{R}$ be a convex function. Then prove that, for all $\alpha \in \mathbb{R}$, the α-level sets are convex. (**2 marks**)
9. Define the Hessian matrix of a function of n variables. (**2 marks**)
10. What is a quadratic programming problem? Give one example. (**1+1 marks**)

Part B (5×16 marks $= 80$ marks)

Instructions

- Question 11 is compulsory in this part. For questions 12 to 15, you can choose to do either part (a) or part (b).
- If you do both parts for a question then marks will be awarded only for the first answered part (which is not crossed-out), even if the solution is not complete.
- Each step must be clearly written and the final answer if it is a numerical value(s) must be boxed and marked separately.

Questions

11. Determine the maximum value of z in the following LPP

$$\begin{array}{rrrrr} \max & z = & c_1x_1 & +c_2x_2 & -2x_3, \\ & x_1 & -x_2 & & \leq 2, \\ & -x_1 & +x_2 & -x_3 & \leq -3, \\ & 2x_1 & -2x_2 & -2x_3 & \leq 2.13, \\ & x_1 & -2x_2 & -2x_3 & \leq 3, \end{array}$$

with $x_1, x_2, x_3 \geq 0$. Assume that the dual of the above LPP has an optimal solution, given by $(1, y_2, y_3, y_4)$. Also determine one value each of c_1 and c_2 and the corresponding values of x_1, x_2, x_3 for which you get the maximum value of z. **(12+4 marks)**

12. (a) Given $f : \mathbb{R}^n \to \mathbb{R}$ is defined by $f(x_1, x_2, \ldots, x_n) = x_1^2 x_2^2 \cdots x_n^2$ and $n \geq 2$. Using the Lagrange multiplier method, find the maximum value of f subject to the constraint

$$g(x_1, x_2, \ldots, x_n) = x_1^2 + x_2^2 + \cdots + x_n^2 = 1.$$

What is the minimum value of f? **(12+4 marks)**

OR

(b) Given $S \subset \mathbb{R}^3$ be an open convex set and $f : S \to \mathbb{R}$ be a differentiable convex function on S. Prove that for all $X, Y \in S$, we have **(16 marks)**

$$f(X) - f(Y) \geq (X - Y) \cdot \nabla f(Y).$$

13. (a) The captain of a cricket team wants to allot four middle order batters. The average runs scored by each batter at these positions are given in the table. Find an arrangement of batters to positions that would give the maximum number of runs.

	A	B	C	D
I	42	35	28	21
II	30	25	20	15
III	30	25	20	15
IV	24	20	16	12

(16 marks)

OR

(b) Derive the Karush-Kuhn-Tucker (KKT) conditions for the inequality constrained optimization problem:

$$\text{Min} \quad f(x)$$

subject to

$$g_i(x) \leq b_i, \quad 1 \leq i \leq m.$$

(16 marks)

14. (a) The payoff matrix of Player A is shown in the table below.

	Player B			
Player A	3	8	4	4
	-7	2	10	2

(8+4+4 marks)

I. Find the optimal solution of the game using the graphical method.
II. Write down the associated LPP with respect to Player A.
III. Write down the associated LPP with respect to Player B.

OR

(b) Derive mathematically the Gomory cut constraint for an MILP. Explain each step with valid mathematical justification. **(16 marks)**

15. (a) Consider the following AILP

$$\text{Max} \quad z = 5x + 2y$$

subject to

$$\begin{aligned} 2x + 2y &\leq 9 \\ 3x + y &\leq 11 \\ x, y &\geq 0 \\ x, y &\in \mathbb{Z}. \end{aligned}$$

Solve the problem using Gomory's cutting plane method and verify graphically that each of the Gomory cut constraint is deleting a part of the feasible solution but not deleting any of the integer feasible points. **(8+8 marks)**

OR

(b) State and prove the strong duality theorem. **(16 marks)**